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Ergodic actions of $S_\mu U(2)$ on C^*–algebras from II_1 subfactors

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Abstract

To a proper inclusion $N \subset M$ of II_1 factors of finite Jones index $[M : N]$, we associate an ergodic C^*–action of the quantum group $S_\mu U(2)$ (or more generally of certain groups $A_o(F)$). The higher relative commutant $N' \cap M_{r-1}$ can be identified with the spectral space of the r-th tensor power $u^{\otimes r}$ of the defining representation of the quantum group. The index and the deformation parameter are related by $-1 < \mu < 0$ and $[M : N] = |\mu + \mu^{-1}|$.

This ergodic action may be thought of as a virtual subgroup of $S_\mu U(2)$, in the sense of Mackey, arising from the tensor category generated by the N–bimodule ${}_N M_N$. μ is negative as ${}_N M_N$ is a real bimodule.

In particular, Wenzl's construction of subfactors from quantum groups indicates a connection, that we do not exploit, between quantum groups at roots of unity and compact quantum groups.

1 Introduction

Ocneanu's approach to subfactor theory is in terms of bimodules and these provide more general tensor categories than those arising from the representations of quantum groups. Not all braid group representations arise from quantum groups, and more are realized naturally within the framework of subfactors of both finite and infinite type. Indeed, in his pioneering work, Jones showed that the categories associated with subfactors always give rise to representations of the braid group [13], [14]. Much work was later dedicated to constructing subfactors from quantum groups. In this respect, Wenzl showed that algebraic quantum groups at roots of unity give rise to subfactors whose bimodule categories have unitary braidings [36], [37], [38].

The bimodule categories derived by passing from quantum groups to subfactors [38] are also considered interesting because they share much of the structure

with the categories of algebraic QFT in low dimensions that also have a unitary braiding (see also [15] and references therein, [17], [11]).

Ocneanu, motivated by the theory of subfactors gave a classification of the subgroups of $SU(k)_l$ at level l [22].

There has been work by Banica, who constructed compact quantum groups from lattices satisfying the axioms of the higher relative commutants [3].

There is a vast literature concerning the problem of constructing a group-type predual of an abstract tensor category. In this respect, our reference list is by no means complete. For permutation symmetric tensor C^*–categories with conjugation the problem was solved in [8], where it was shown that the predual is a compact group.

The aim of this note is to associate with the category of bimodules of an inclusion $N \subset M$ a 'compact predual'. Now the bimodule category of a II_1 inclusion is generically *not* embedable into the category of Hilbert spaces, and hence is not the category of representations of a compact quantum group. Thus the notion of a compact group-type predual will need to be understood in a wider sense.

To this end we shortly review the original connection between subfactors and group theory. It came from II_1 subfactors $N \subset M$ of finite index where the associated Jones projections give rise to a representation of the infinite braid group which factors through the Temperley-Lieb relations, showing that the bimodule category of the inclusion contains a deformation of the subcategory of the representation category of $SU(2)$ generated by the flip map, see, e.g., [10]. As we are not dealing with a representation of the symmetric group, the group involved has a noncommutative structure, i.e. is a quantum group.

Our starting point is the observation that while ${}_NM_N$ is a selfconjugate *real* bimodule, the defining representation of $SU(2)$ is selfconjugate and *pseudoreal*.

However, the defining representation of the compact quantum group $S_\mu U(2)$ is real provided the deformation parameter μ is negative (see, e.g., [26]). We shall see that if μ is suitably chosen, the tensor C^*–subcategory of $\mathrm{Rep}(\mathrm{S}_\mu\mathrm{U}(2))$ generated by the fundamental representation embeds into the $SU(2)$–like category contained in the bimodule tensor category $\mathcal{T}$ generated by ${}_NM_N$. A natural question is therefore whether we can find a 'subgroup' of $S_\mu U(2)$ acting as a predual of $\mathcal{T}$, where the notion of subgroup needs to be understood in a wide sense, resembling, perhaps, the virtual subgroups of Mackey in classical ergodic theory [19], [20] but in a compact and noncommutative setting.

It is therefore natural to expect such virtual subgroups to appear as ergodic actions of $S_\mu U(2)$ on unital C^*–algebras. Here ergodicity is meant in the usual sense of lacking nontrivial fixed point elements, and a subfactor indeed determines such an ergodic action.

As Banica showed, the representation category of a compact quantum group G does not determine G uniquely. In fact, the Van Daele–Wang groups $A_o(F)$ with $\mathrm{Tr}F^*F = |\mu + \frac{1}{\mu}|$ and $F\overline{F} = \pm I$ have representation category tensor *–equivalent to that of $S_\mu U(2)$, with μ respectively negative or positive [1], [2]. [1]

[1] Wang and Van Daele denote $A_u(F)$ by $A_u(Q)$ where $Q = F^*F$ and $A_o(F)$ by $B_u(Q)$,

Consequently we expect ergodic actions of all such quantum groups $A_o(F)$.

In this note, applying results recently obtained in [27], we construct ergodic C^*–actions of $A_o(F)$, and in particular of $S_\mu U(2)$, canonically associated with the bimodule ${}_N M_N$ of a proper, finite index, II_1 inclusion $N \subset M$. This action has the following features: the deformation parameter depends on the index in such a way that the index equals the quantum dimension of the defining representation of $S_\mu U(2)$: $[M : N] = |\mu + \mu^{-1}|$ and $-1 < \mu < 0$. In particular, from the pivotal result of Jones restricting the values of the index [13], only the values $4\cos^2 \pi/m$ and ≥ 4 of the quantum dimension of $S_\mu U(2)$ can possibly arise from our examples.

Taking into account Wenzl's work on constructing subfactors from algebraic quantum groups, we pass through subfactors and connect quantum groups at roots of unity to ergodic actions of compact quantum groups whose quantum dimension depends on roots of unity. Thus quantum groups at roots of unity might appear as virtual quantum subgroups of compact quantum groups.

In contrast with the examples of [4], the quantum multiplicity of $u^{\otimes r}$ in the ergodic action, with u the defining representation of $A_o(F)$, takes its lower bound, the integral multiplicity, in turn given by the dimension of the higher relative commutant $N' \cap M_{r-1}$. Moreover the spectral space corresponding to $u^{\otimes r}$ with its Hilbert space structure can be identified with the higher relative commutant $N' \cap M_{r-1}$, with inner product coming from the Markov trace.

Furthermore these ergodic actions are not, in general, embedable into the translation action. We shall show this for non-integral values of the index of an extremal and amenable inclusion. The proof relies on Popa's work [30]. In particular, the C^*–algebra involved cannot be commutative.

We show in [28] that the results of this paper have analogues when subfactors are replaced by solutions of the conjugate equations. The groups involved are $A_u(F)$ and $A_o(F)$.

Following Mackey's point of view, a virtual subgroup should actually exhibit typical properties of a closed subgroup. We shall develop this matter in a subsequent paper, where we will consider representations of virtual subgroups corresponding to ergodic actions arising from general tensor C^*–categories with conjugation [9].

The paper is organized as follows. In section 2 we recall the main invariants of ergodic C^*–actions and the duality theorem of [27].

In section 3 we recall Ocneanu's bimodules associated with an inclusion of II_1 factors with finite Jones index and we show that extremal and amenable inclusions in the sense of [23], [30] give rise to categories which generically are not embedable into the category of Hilbert spaces. When they are, only quantum groups with coinvolutive coinverses appear. Hence Ocneanu's categories are, in this respect, rather different from the representation categories of compact quantum groups, which are embedded in Hilbert spaces by construction.

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where $Q = F^*$, [31].

In section 4 we state our main results: the inclusion gives rise to ergodic C^*–actions of the Van Daele-Wang compact quantum group $A_o(F)$ associated with an invertible matrix $F \in M_n(\mathbb{C})$.

The matrix F and the integer n are required to satisfy the following properties: $2 \leq n \leq [M : N]$, $F\overline{F} = I$. When $n = 2$, we find ergodic actions of $S_\mu U(2)$ for a deformation parameter related to $[M : N]$ as described above.

If $[M : N]$ is an integer, p, then we find an ergodic C^*–action of the Kac compact quantum group $A_o(I_p)$. We show that in this case the unique invariant state is a trace.

Section 5 is devoted to the proofs of the results. We conclude the paper with the necessary computations that yield a presentation of the dense *–subalgebra of spectral elements by generators and relations, in terms of the higher relative commutants $N' \cap M_r$.

2 Preliminaries on ergodic C^*–actions

Consider a unital C^*–algebra $\mathcal{C}$ and a compact quantum group $G = (\mathcal{Q}, \Delta)$, with coproduct $\Delta : \mathcal{Q} \to \mathcal{Q} \otimes \mathcal{Q}$ [41]. An *action* of G on $\mathcal{C}$ is a unital *–homomorphism $\delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{Q}$ satisfying the group representation property:

$$\iota \otimes \Delta \circ \delta = \delta \otimes \iota \circ \delta,$$

and the nondegeneracy property requiring $\delta(\mathcal{C})I \otimes \mathcal{Q}$ to be dense in $\mathcal{C} \otimes \mathcal{Q}$. The *spectrum* of δ, $\mathrm{sp}(\delta)$, is defined to be the set of all representations u of G for which there is a faithful linear map $T : H_u \to \mathcal{C}$ intertwining the representation u with the action δ:

$$\eta \circ T = T \otimes \iota \circ u.$$

In other words, if u_{ij} are the coefficients of u corresponding to some orthonormal basis of H, we are requiring the existence of linearly independent elements $c_1, \ldots, c_d \in \mathcal{C}$, with d the dimension of u, transforming like u under the action: $\eta(c_i) := \sum_j c_j \otimes u_{ji}$. The linear span of all the c_i's, denoted $\mathcal{C}_{\mathrm{sp}}$, as u varies in the spectrum, is a dense *–subalgebra of $\mathcal{C}$ [29].

The action δ is called *ergodic* if the fixed point algebra

$$\mathcal{C}^\delta = \{c \in \mathcal{C} : \delta(c) = c \otimes I\}$$

reduces to the complex numbers: $\mathcal{C}^\delta = \mathbb{C}I$. The simplest example of an ergodic action is the translation action of G on $\mathcal{C} = \mathcal{Q}$ with $\delta = \Delta$.

If an action δ is ergodic, spectral multiplets can be organized to form Hilbert spaces. In fact, for any representation u, consider the space

$$L_u := \{T : H_u \to \mathcal{C}, \delta \circ T = T \otimes \iota \circ u\}.$$

If $S, T \in L_u$, $< S, T >:= \sum_i T(\psi_i)S(\psi_i)^*$, with (ψ_i) an orthonormal basis of H_u, is an element of the fixed point algebra $\mathcal{C}^\delta$, and therefore a complex number. L_u is known to be finite dimensional, and therefore a Hilbert space

with the above inner product. This Hilbert space is nonzero precisely when u contains a subrepresentation $v \in \mathrm{sp}(\delta)$. In particular, for an irreducible u, the conditions $u \in \mathrm{sp}(\delta)$ and $L_u \neq 0$ are equivalent. The dimension of L_u is called the *multiplicity* of u and denoted $\mathrm{mult}(u)$.

The complex conjugate vector space $\overline{L_u}$, endowed with the conjugate inner product

$$< \overline{S}, \overline{T} >:=< T, S >= \sum_i S(\psi_i)T(\psi_i)^*,$$

is called the *spectral space* associated with u.

If for example δ is the translation action on $\mathcal{Q}$, any $\psi \in H_u$ defines an element of L_u by

$$T_\psi(\psi') := \psi^* \otimes Iu(\psi').$$

Hence the spectral space $\overline{L_u}$ can be identified with H_u through the unitary map

$$\psi \in H_u \to \overline{T_\psi} \in \overline{L_u}.$$

In [4] the authors introduce a new numerical invariant, the *quantum multiplicity* $m(u)$ of the representation u in the following way. If $j : H_u \to H_{\overline{u}}$ is an antilinear invertible defining a conjugate (unitary) representation $\overline{u}$ of u by $\overline{u}_{\phi,j\psi} = (u_{j^*\phi,\psi})^*$, then there is an associated antilinear $J : L_u \to L_{\overline{u}}$ by $J(T)(\phi) := T(j^{-1}(\phi))^*$ with inverse $J^{-1} : L_{\overline{u}} \to L_u$ given by $J^{-1}(S)(\psi) = S(j(\psi))^*$. If u is irreducible, $m(u)^2 := \mathrm{Trace}(JJ^*)\mathrm{Trace}((JJ^*)^{-1})$. One has:

$$\mathrm{mult}(u) \leq m(u) \leq d(u),$$

an inequality which strenghtens the inequality $\mathrm{mult}(u) \leq d(u)$ previously obtained by Boca [6], in turn generalizing HLS theorem [12] $\mathrm{mult}(u) \leq \dim(u)$ in the group case. If u is reducible, we define $m(u)$ as the infimum of all the above trace values, when j ranges over all possible solutions of the conjugate equations. Then the inequality

$$\dim(L_u) \leq m(u) \leq d(u) \tag{2.1}$$

now holds for all representations u. Notice that $m(u)$ takes the smallest possible value $\dim(L_u)$ precisely when for some j the associated J is a scalar multiple of an antiunitary. The actions we shall construct in this paper satisfy this property. Examples of ergodic actions of $S_\mu U(2)$ where $\dim(u) < \mathrm{mult}(u) < m(u) = d(u)$ have been constructed in [4].

It has been shown in [27] that ergodic actions of compact quantum groups admit a duality theory that resembles the Tannaka-Krein duality theory: It is possible to associate a dual object with an ergodic G–action from which the G–action on the maximal completion of $\mathcal{C}_{\mathrm{sp}}$ can be reconstructed. Furthermore all possible dual objects have been characterized.

The spectral functor

$$\overline{L} : \mathrm{Rep}(G) \to \mathcal{H}$$

associated with an ergodic G–action is the functor from the category of representations of G to the category $\mathcal{H}$ of Hilbert spaces. This functor is defined as $u \to \overline{L_u}$ on objects and as follows on arrows.

If $A \in (u, v)$ and $T \in L_v$ then $T \circ A : H_u \to \mathcal{C}$ lies in L_u. Hence if we identify $\overline{L_u}$ canonically with the dual vector space of L_u, any arrow $A \in (u, v)$ in $\mathrm{Rep}(G)$ induces a linear map $\overline{L}_A \in (\overline{L_u}, \overline{L_v})$ by

$$\overline{L}_A : \varphi \in \overline{L}_u \to (T \in L_v \to \varphi(T \circ A)) \in \overline{L_v}.$$

Taking into account the tensor C^*–category structure of $\mathrm{Rep}(G)$ and $\mathcal{H}$ one can see that $\overline{L}$ becomes a *–functor, but *not* a tensor *–functor, in general.

As far as the tensor structure of $\overline{L}$ is concerned, for $u, v \in \mathrm{Rep}(G)$, the tensor product Hilbert space $\overline{L_u} \otimes \overline{L_v}$ is in general just a subspace of $\overline{L_{u \otimes v}}$, in the sense that there is a natural isometric inclusion

$$\lambda_{u,v} : \overline{L_u} \otimes \overline{L_v} \to \overline{L_{u \otimes v}}$$

identifying any simple tensor $\overline{S} \otimes \overline{T}$ with the complex conjugate of the element of $L_{u \otimes v}$ defined by

$$\psi \otimes \phi \in H_u \otimes H_v \to S(\psi)T(\phi).$$

The dual of the action is the pair $(\overline{L}, \lambda)$ consisting of the functor $\overline{L}$ and all the inclusions $\lambda_{u,v}$.

The maximal C^*–completion of $\mathcal{C}_{\mathrm{sp}}$, together with the extended G–action, can be reconstructed from the dual $(\overline{L}, \lambda)$.

The main result of [27] is an algebraic characterization of the set of all duals of ergodic actions $(\overline{L}, \lambda)$ among all *–functors

$$\tau : \mathrm{Rep}(G) \to \mathcal{H}$$

endowed with isometries $\tilde{\tau}_{u,v} : \tau_u \otimes \tau_v \to \tau_{u \otimes v}$. All pairs $(\tau, \tilde{\tau})$ satisfying properties (3.1)–(3.6) in [27] have been shown to arise as the dual of an ergodic G–action. Such functors are called *quasitensor*. More precisely, an ergodic C^*–action of G on a unital C^*–algebra $\mathcal{C}$ can be constructed by duality from a quasitensor *–functor $(\tau, \tilde{\tau}) : \mathrm{Rep}(G) \to \mathcal{H}$. Once the ergodic action has been constructed, the pair $(\tau, \tilde{\tau})$ can be identified with the dual object of that action.

3 Ocneanu's category from a II_1 inclusion

Consider an inclusion of II_1 factors $N \subset M$ of finite Jones index $[M : N]$ [13] and denote the trace-preserving conditional expectation by $E : M \to N$. Let

$$N \subset M \subset M_1 \subset M_2 \ldots$$

be the Jones tower of II_1 factors. We denote the r-th Jones projection derived from the trace-preserving conditional expectation $E_{r-1} : M_{r-1} \to M_{r-2}$ by $e_r \in L^2(M_{r-1})$, where we have set $M_0 = M$, $M_{-1} = N$, $E_0 = E$. Recall that

$M_r := M_{r-1}e_rM_{r-1}$ and that $E_r(e_r) = [M : N]^{-1}$. Also recall that the algebras $N' \cap M_r$, usually called the higher relative commutants, are finite dimensional and

$$\dim(N' \cap M_{r-1}) \leq [M : N]^r, \quad r \geq 0.$$

The main relations are the following. For $r \geq -1$,

$$[M_r, e_{r+2}] = 0, \tag{3.1}$$

$$e_{r+1}me_{r+1} = E_r(m)e_{r+1}, \quad m \in M_r, \tag{3.2}$$

implying the Jones projection relations:

$$e_ie_j = e_je_i, \quad |i - j| \geq 2, \tag{3.3}$$

$$e_ie_{j\pm1}e_i = [M : N]^{-1}e_i. \tag{3.4}$$

We review the well known construction of the Jones tower, M_r, $r \geq 0$, in terms of Ocneanu's bimodules [21], [23], [24], [5], [16]. Regard M as a right Hilbert N–module with N–valued inner product

$$< m, m' >:= E(m^*m'), \quad m, m' \in M.$$

Since the index is finite, M is finitely generated over N. Left multiplication on M by elements of N makes M into a Hilbert bimodule in the sense considered in [25]. Therefore we can take tensor powers of M over N and get further Hilbert bimodules. When no confusion arises, this tensor product will be simply denoted by $\otimes$. As N–bimodules:

$$M \otimes M \simeq M_1.$$

Therefore iteratively, for $r = 1, 2, \ldots,$

$$M_{r-1} \simeq M^{\otimes r}. \tag{3.5}$$

Consider the category $\mathcal{T}_M$ with objects the N–bimodules $M^{\otimes r}$, $r \geq 0$, and arrows the bimodule mappings. This is a tensor C^*–category in the sense of [8], with tensor product structure on arrows naturally induced by the tensor product of Hilbert bimodules. The arrow space $(\iota, M^{\otimes r})$ can be identified, as a vector space, with $N' \cap M_{r-1}$. In particular, $(\iota, M) \simeq N' \cap M \neq 0$ and is one-dimensional precisely when the inclusion is irreducible. Therefore $\mathcal{T}_M$, as a tensor C^*–category, is determined by an isometry $S \in (\iota, M)$ and its subcategory of arrow spaces $(M^{\otimes r}, M^{\otimes r})$ of grade zero. Identifying the N–bimodules $M^{\otimes r} \simeq M_{r-1}$, the algebra $\mathcal{L}_N(M^{\otimes r})$ of right N–module maps can be identified with Jones's basic construction associated with the inclusion $N \subset M_{r-1}$, which, in turn, can be identified with M_{2r-1} as an algebra. Taking into account the *–involution arising from the tensor power inner product of $M^{\otimes r}$, it is well known that we have an isomorphism of C^*–algebras $\mathcal{L}_N(M^{\otimes r}) \simeq M_{2r-1}$. Therefore as C^*–algebras,

$$(M^{\otimes r}, M^{\otimes r}) \simeq N' \cap M_{2r-1}.$$

Only terms of the Jones tower with odd indices appear as we started with the bimodule ${}_NM_N$ rather than $\sigma := {}_MM_N$ or $\overline{\sigma} := {}_NM_M$. One has ${}_NM_N \simeq \overline{\sigma}\otimes_M\sigma$.

We next give a result showing that the tensor categories $\mathcal{T}_M$ arising from *amenable* inclusions in the sense of [30], are generically not embedable into categories of Hilbert spaces. In nongeneric cases, where $\mathcal{T}_M$ is embedable, it must generate the category of representations of a compact quantum group with involutive coinverse. Hence inclusions of factors provide many categories that cannot arise from representations of compact quantum groups.

3.1 Theorem *Let $N \subset M$ be a finite index, extremal and amenable inclusion of II_1 factors. If $[M : N]$ is not an integer then the tensor C^*–category $\mathcal{T}_M$ cannot be embedded into the category of Hilbert spaces. Conversely, if $[M : N]$ is an integer and if $\mathcal{T}_M$ is embedable then the Hilbert space corresponding to ${}_NM_N$ has dimension $[M : N]$ and $\mathcal{T}_M$ is isomorphic to the representation category of a compact quantum subgroup of the compact Kac quantum group $A_o(I_{[M:N]})$ where $I_{[M:N]}$ is the identity matrix of size $[M : N]$.*

Proof Popa shows that under the amenability assumption,

$$[M : N] = \lim_r \dim(N' \cap M_r)^{1/r}$$

(see Theorem 4.4.1(3) in [30]). Therefore if $[M : N]$ is not an integer, for sufficiently large r,

$$\dim(\iota, M^{\otimes r}) = \dim(N' \cap M_{r-1}) > i^r$$

where i is the integral part of the index. If $\mathcal{T}_M$ were embedable and if n denotes the dimension of the Hilbert space corresponding to M then we must have $n \leq [M : N]$. Hence $n \leq i$. Furthermore for all r we should also have

$$\dim(\iota, M^{\otimes r}) \leq n^r \leq i^r,$$

a contradiction. Conversely, assume that $[M : N] = i$ and $\mathcal{T}_M$ embedable. If the Hilbert space corresponding to M had dimension $n < i$ then

$$i = \lim_r \dim(\iota, M^{\otimes r})^{1/r} \leq n < i,$$

again a contradiction. Therefore the dimension of the Hilbert space H corresponding to M is uniquely determined by the index. The intertwiner R needs then to correspond to an element $R' \in H \otimes H$ with $\|R'\|^2 = [M : N] = \dim(H)$. Since R' makes H into a pseudoreal object, the analysis in [4] shows that $\|R'\|^2 = \dim(H)$ is possible only if $R' = \sum_i e_i \otimes e_i$ for some orthonormal basis of H, and the proof is complete.

Known classes of extremal and *strongly amenable* [30] (hence amenable) inclusions are given by: subfactors N of the hyperfinite II_1 factor R with $[R : N] \leq 4$ or of finite index and finite depth, e.g. the Jones subfactors [10] and Wenzl subfactors arising from representations of the Hecke algebras of

type A, B, C, D at roots of unity [36], [37] (these provide non-embedable categories as the index is not integral); subfactors arising from actions of compact groups G on R and finite dimensional unitary irreducible G–representations[10], [35] (these, on the contrary, provide embedable categories). All these examples, together with a complete list of references are discussed in [30].

Remark A notion of amenability for an object ρ of a tensor C^*–category is introduced in [18]. It implies $d(\rho) = \lim_r \dim(\rho^r, \rho^r)^{1/2r}$. As here, it is shown that $\mathcal{T}_\rho$ is not embedable unless $d(\rho)$ is an integer and that when it is then $d(\rho) = \dim H(\rho)$.

4 Statement of the results

4.1 Theorem *Let $N \subset M$ be a proper inclusion of II_1 factors with finite index. For any integer $2 \leq n \leq [M:N]$, let $F \in M_n(\mathbb{C})$ be an invertible matrix satisfying $F\overline{F} = I$, $\quad Trace(F^*F) = [M:N]$. Then*

a) *there is an ergodic action of $A_o(F)$ on a unital C^*–algebra $\mathcal{C}$ with spectral spaces $\overline{L}_{u^{\otimes r}} = N' \cap M_{r-1}$, $r \geq 0$, with inner product defined by the restriction of the normalized trace on M_{r-1}. One has:*

$$m(u^{\otimes r}) = dim(N' \cap M_{r-1}),$$

where u is the defining representation of $A_o(F)$.

b) *In particular, if we choose $n = 2$ we get an action of $S_{-\mu}U(2)$ with $0 < \mu < 1$ determined by $\mu + \mu^{-1} = [M:N]$.*

c) *If $[M:N]$ is an integer, say p, then we get an ergodic C^*–action of the Kac compact quantum group $A_o(I_p)$. In this case the unique invariant state is a trace.*

4.2 Corollary *If the inclusion $N \subset M$ is extremal and amenable in the sense of [30] and if $[M:N]$ is not an integer then the ergodic action of $A_o(F)$ as above is not embedable into the translation action on $A_o(F)$.*

Proof By [30],

$$\lim \dim(\overline{L_{u^{\otimes r}}})^{1/r} = \lim \dim(N' \cap M_{r-1})^{1/r} = [M:N]. \tag{4.1}$$

Hence if $[M:N]$ is not an integer, $\dim(\overline{L_{u^{\otimes r}}}) > n^r$ for r large enough. Thus the action cannot be embedable in the translation action, as this would imply $\overline{L_{u^{\otimes r}}} \subset H_{u^{\otimes r}}$ for all r.

Example 1 As mentioned in the Introduction, consider the Jones subfactor $R_\beta \subset R$ of the hyperfinite II_1 factor with index $\beta = 4\cos^2 \pi/m$, with $m \geq 4$. Here ${R_\beta}' \cap R_{r-1}$ is the algebra $B_{\beta,r}$ generated by the Jones projections $e_1, \ldots, e_{r-1}$. It carries a unitary representation of the braid group $\mathbb{B}_r$ as we are

in the case $\beta < 4$, see, [13] or [10] (hence the tensor C^*–category generated by the Hilbert bimodule ${}_N M_N$ has a unitary braiding, as we find such representations in the C^*–algebras $(M^{\otimes r}, M^{\otimes r}) \simeq {R_\beta}' \cap R_{2r-1}$). If we apply the previous theorem, we deduce that $B_{\beta,r}$, regarded as a Hilbert space with inner product defined by its Markov trace, does arise as the spectral space $\overline{L_{u^{\otimes r}}}$ of an ergodic C^*–action of $S_{-\mu}U(2)$, with $\mu + \mu^{-1} = \beta$.

Example 2 In more generality, let $N \subset M$ be the II_1 subfactor arising from quantum groups at roots of unity as in [38]. The higher relative commutants $N' \cap M_{r-1}$ are there shown to correspond to the arrow spaces of the fusion tensor category of the quantum group (Theorem 4.4 in [38]). These spaces, by the previous theorem, again arise as spectral spaces of ergodic actions of compact quantum groups. Furthermore the quantum dimension of the compact quantum group in question depends on the roots of unity. This connection will not be exploited further in this paper.

For convenience, we next give a presentation by generators and relations of the ergodic C^*–algebra $\mathcal{C}$ in terms of the higher relative commutants. This will be deduced from a simpler presentation in terms of natural relations in Ocneanu's tensor C^*–category of Hilbert bimodules that will appear in the course of the proof in the next section, (cf. relations (5.1)–(5.3) and Theorem 5.2).

Before stating the result we need some notation. Set $H := \mathbb{C}^{\times n}$, $j := Fc$, where $c : H \to H$ is the antiunitary fixing the canonical basis of H, and

$$R_u := \sum \psi_k \otimes j\psi_k,$$

where (ψ_k) is any orthonormal basis of H. We introduce certain reduced words in the algebra generated by the Jones projections. Set $\lambda := [M : N]^{1/2}$ and for nonnegative integers k, r, s, define elements $p^{(k)}_{r,s} \in M_{k+r+s-1}$ by

$$p^{(k)}_{0,s} = p^{(k)}_{r,0} := I,$$

and, for $r, s \geq 1$,

$$p^{(k)}_{r,s} := \lambda^{rs}(e_{r+k}e_{r+k-1}\dots e_{1+k})\dots(e_{r+k+s-1}e_{r+k+s-2}\dots e_{s+k}). \tag{4.2}$$

We shall simply write $p_{r,s}$ for $p^{(0)}_{r,s}$. By [24], $p_{r,r}$ reduces to a scalar multiple of the Jones projection associated to $N \subset M_{r-1}$.

In the next result, in order to avoid confusion with the tensor products, we shall denote the r-th tensor powers of H and u by H^r and u^r respectively.

4.3 Theorem *The C^*–algebra $\mathcal{C}$ is obtained by completing the *–subalgebra with generators $\overline{T} \otimes \xi$, $T \in N' \cap M_{r-1}$, $\xi \in H^r$, $r = 0, 1, 2, \dots$, and relations, where $\xi' \in H^s$, $\xi_1, \dots, \xi_r \in H$, $\eta \in H^{r+s+2}$, $\eta' \in H^{r+s}$,*

a) $(\overline{T} \otimes \xi)(\overline{T'} \otimes \xi') = \overline{T p_{r,s} T'} \otimes \xi\xi'$,

b) $(\overline{T} \otimes \xi_1 \dots \xi_r)^* = \overline{T^*} \otimes j\xi_r \dots j\xi_1$,

for $r \geq s$:

c) $\overline{S} \otimes (1_{u^r} \otimes R_u^* \otimes 1_{u^s}\eta) = \lambda \overline{S p_{r-s,2}^{(2s)}} \otimes \eta$,

c') $\overline{S'} \otimes (1_{u^r} \otimes R_u \otimes 1_{u^s}\eta') = \lambda \overline{E_{r+s}E_{r+s+1}(S'(p_{r-s,2}^{(2s)})^*)} \otimes \eta'$,

for $r < s$:

d) $\overline{S} \otimes (1_{u^r} \otimes R_u^* \otimes 1_{u^s}\eta) = \lambda \overline{p_{2,s-r}^{(2r)} S} \otimes \eta$,

d') $\overline{S'} \otimes (1_{u^r} \otimes R_u \otimes 1_{u^s}\eta') = \lambda \overline{E_{r+s}E_{r+s+1}((p_{2,s-r}^{(2r)})^* S')} \otimes \eta'$,

in the maximal C^*–norm. The $A_o(F)$–action β is uniquely defined by

$$\beta(\overline{T} \otimes \xi) = \overline{T} \otimes u^{\otimes r}(\xi), \quad T \in N' \cap M_{r-1}, \xi \in H^r,$$

where u is the defining representation of $A_o(F)$ on H.

5 Proof of the results

We shall refer to [8] for the definition of an abstract tensor C^*–category $\mathcal{T}$. The tensor unit object will be denoted ι. We shall always assume $(\iota, \iota) = \mathbb{C}$. An object ρ will be called real (or pseudoreal) if there is an $R \in (\iota, \rho^2)$ satisfying $R^* \otimes 1_\rho \circ 1_\rho \otimes R = 1_\rho$ (or $R^* \otimes 1_\rho \circ 1_\rho \otimes R = -1_\rho$).

5.1 Theorem *Let ρ be a real or pseudoreal object of $\mathcal{T}$ defined by $R \in (\iota, \rho^2)$ with $\|R\|^2 \geq 2$. For any integer $2 \leq n \leq \|R\|^2$ let $F \in M_n(\mathbb{C})$ be an invertible matrix such that $F\overline{F} = \pm I$ and Trace$(F^*F) = \|R\|^2$. Then there is an ergodic action of $A_o(F)$ on a unital C^*–algebra $\mathcal{C}$ with spectral functor $\overline{L}$, where $\overline{L}_{u^{\otimes r}} = (\iota, \rho^r)$ and $\overline{L}_{\sum_k \psi_k \otimes F\psi_k}$ is left compositon by R. In particular, for $n = 2$ we get an action of $S_\mu U(2)$ for a nonzero $-1 < \mu < 1$ determined by $|\mu + \mu^{-1}| = \|R\|^2$ and $\mu > 0$ if ρ is pseudoreal and negative otherwise.*

The above theorem was proved in [27] . We shall outline the proof for convenience.

Outline of proof. One first shows that the tensor *–subcategory generated by an arrow R making ρ real (or pseudoreal) is uniquely determined by the quantum dimension $d = \|R\|^2$. If F is chosen as indicated, it gives rise to a realization of this category in the category of Hilbert spaces which generates the representation category of $A_o(F)$. Thus we have a tensor *–functor from the full tensor subcategory of Rep$(A_o(F))$ generated by the fundamental representation u to $\mathcal{T}_\rho$ taking u to ρ and the basic intertwiner $\sum_k \psi_k \otimes F\psi_k$ to R. We next apply the duality theorem in [27] (cf. Sect. 2) to the quasitensor functor $u^{\otimes r} \to (\iota, \rho^r)$. That theorem yields a presentation by generators and relations of the dense linear space (in fact a *–subalgebra) $\mathcal{C}_{\rm sp}$ generated by the spectral elements $T^* \otimes \xi$ with relations:

$$(T^* \otimes \xi)(T'^* \otimes \xi') = T^* \otimes T'^* \otimes \xi\xi', \tag{5.1}$$

$$(T^* \otimes \xi)^* = \rho(R^*_{u^r}) \circ 1_{\overline{\rho_{u^r}}} \otimes T \otimes j_{u^r}\xi, \tag{5.2}$$

$$T^*\rho(A) \otimes \xi' = T^* \otimes A\xi', \tag{5.3}$$

where $T \in (\iota, \rho^r)$, $T' \in (\iota, \rho^s)$, $\xi \in H^r$, $\xi' \in H^s$, $A \in (u^s, u^r)$, and j_{u^r} defines a solution R_{u^r}, $\overline{R}_{u^r}$ of the conjugate equations for the r-th tensor power u^r of the defining representation u.

Finiteness of the Jones index implies that $\mathcal{T}_M$ has conjugates. We start by recalling some facts on module bases for M_N from [23]. A basis for M, as a right Hilbert module, is a finite set of elements (u_i) in M such that $\sum_i u_i E(u_i^* m) = m$ for all $m \in M$. We shall refer to (u_i) as a Pimsner-Popa basis. Such bases exist as M_N is finitely generated. If (u_i) is a Pimsner-Popa basis then $\sum_i u_i e_1 u_i^* = I$ in M_1. We have: $\sum_i u_i u_i^* = [M : N]$.

5.2 Theorem *${}_N M_N$ is a real object of the category $\mathcal{T}_M$. A solution of the conjugate equations for M is given by*

$$R = \overline{R} = \sum_i u_i \otimes u_i^*, \tag{5.4}$$

where (u_i) is a Pimsner-Popa basis for M. One has $\|R\|^2 = [M : N]$.

Proof As we have a tensor product over N it is easily checked that R is independent of the choice of the Pimsner–Popa basis. On the other hand for any unitary $u \in N$, (uu_i) is another Pimsner–Popa basis, hence $uRu^* = R$, showing that R is an intertwiner in $\mathcal{T}_M$. For $m \in M$,

$$R^* \otimes 1_M \circ 1_M \otimes R(m) = \sum_i R^* \otimes 1_M(m \otimes u_i \otimes u_i^*) =$$

$$\sum_{i,j} < u_j \otimes u_j^*, m \otimes u_i > u_i^* = \sum_{i,j} < u_j^*, E(u_j^* m)u_i > u_i^* =$$

$$\sum_{i,j} E(u_j E(u_j^* m)u_i)u_i^* = \sum_i E(mu_i)u_i^* = m.$$

We also have

$$\|R\|^2 = R^*R = \sum_{i,j} < u_i \otimes u_i^*, u_j \otimes u_j^* >=$$

$$\sum_{i,j} E(u_i E(u_i^* u_j)u_j^*) = \sum_j E(u_j u_j^*) = [M : N].$$

Applying Theorem 5.1 to Ocneanu's category $\mathcal{T}_M$ and the intertwiner $R = \sum u_i \otimes u_i^*$, we get the desired ergodic action of $A_o(F)$ on a unital C^*–algebra with spectral spaces $\overline{L}_{u^{\otimes r}} = (\iota, M^{\otimes r}) \simeq N' \cap M_{r-1}$. It remains to identify the inner products on the spaces $(\iota, M^{\otimes r})$ arising from the category $\mathcal{T}_M$ and

the algebraic presentation of $\mathcal{C}_{\rm sp}$ in terms of the higher relative commutants $N' \cap M_{r-1}$.

Now the inner product of $(\iota, M^{\otimes r})$ is the restriction of the N–valued inner product of $M^{\otimes r}$ which, through the unitary identification $M^{\otimes r} \simeq L^2(M_{r-1})$ as N–N–correspondences in the sense of Connes [7], arises from the normalized trace of M_{r-1} (see, e.g., Prop. 3.1 in [5]).

For later use, we shall need explicit Hilbert N–bimodule unitaries from $M^{\otimes r}$ to M_{r-1}. Consider the N–bimodule isomorphism between $M_{r+1} := M_r e_{r+1} M_r$ and $M_r \otimes_{M_{r-1}} M_r$, given by

$$m \otimes m' \to \lambda m e_{r+1} m', \quad m, m' \in M_r.$$

Regard M_r as a M_{r-1}–Hilbert bimodule with inner product defined by the normalized conditional expectation $E_r : M_r \to M_{r-1}$. Then the tensor product of the M_{r-1}–valued inner products on $M_r \otimes_{M_{r-1}} M_r$, corresponds, under the above isomorphism U to the inner product induced by the (normalized) conditional expectation $E_r E_{r+1} : M_{r+1} \to M_{r-1}$ by

$$< S, T >= E_r E_{r+1}(S^* T), \quad S, T \in M_{r+1}.$$

Iterating this procedure (recall (3.5)), leads to canonical isomorphisms of N–bimodules $\Gamma_r : M^{\otimes r} \to M_{r-1}$ transforming the tensor product of N–valued inner products into the inner product induced by the conditional expectation

$$E_{(r)} := E E_1 \dots E_{r-2} E_{r-1} : M_{r-1} \to N. \tag{5.5}$$

$\Gamma_r : M^{\otimes r} \simeq M_{r-1}$ is obtained in the following way. First replace each factor M occurring in 2nd to $r-1$th position in $M^{\otimes r}$ by $M \otimes_M M$, giving $2r-2$ factors tensored alternately over N and M. Thus

$$M^{\otimes r} = (M \otimes_N M)^{\otimes_M r-1}.$$

Finally use the isomorphism $M \otimes_N M \simeq M_1$ giving by iteration

$$M^{\otimes r} \simeq M_1^{\otimes_M r-1} \simeq M_2^{\otimes_{M_1} r-2} \simeq \dots \simeq M_{r-2} \otimes_{M_{r-3}} M_{r-2} \simeq M_{r-1}.$$

The resulting isomorphism Γ_r is

$$m_1 \otimes \dots \otimes m_r \to \lambda^{<r>} m_1 e_1 m_2 e_2 e_1 m_3 \dots m_{r-1} e_{r-1} \dots e_1 m_r, \tag{5.6}$$

where $< r >:= (r-1) + (r-2) + \dots + 1 = r(r-1)/2$. Summarizing, one has the following result.

5.3 Proposition *Under the N–bimodule isomorphisms $\Gamma_r : M^{\otimes r} \to M_{r-1}$, the N–valued tensor power inner product on $M^{\otimes r}$ corresponds to the inner product on M_{r-1} defined by the trace-preserving conditional expectation $E_{(r)} : M_{r-1} \to N$:*

$$< S, T >:= E_{(r)}(S^* T), \quad S, T \in M_{r-1}.$$

In particular, the Hilbert space structure of $(\iota, M^{\otimes r})$ *defined by the category* $\mathcal{T}_M$ *corresponds, under the restriction of* Γ_r*, to the inner product of* $N' \cap M_{r-1}$ *defined by the restriction of the normalized trace on* M_{r-1}*.*

We need to establish the algebraic presentation of the dense spectral *–subalgebra $\mathcal{C}_{\rm sp}$. From [27] (cf. Theorem 5.1 and its proof) applied to $\mathcal{T} = \mathcal{T}_M$, Ocneanu's tensor C^*–category associated with the inclusion $N \subset M$ as in section 3, and to the tensor *–functor from the full tensor C^*–subcategory of $\mathrm{Rep}(A_o(F))$ generated by the defining representation u to $\mathcal{T}_M$ and taking u to ${}_N M_N$ and the basic intertwiner $\sum_k \psi_k \otimes F\psi_k$ to $R = \sum u_i \otimes u_i^*$, we know that an algebraic presentation of $\mathcal{C}_{\rm sp}$ is given by generators $T^* \otimes \xi$ with $T \in (\iota, M^{\otimes r})$, $\xi \in H^r$ subject to the relations (5.1)–(5.3).

We start by computing the *–involution. Starting from the antilinear intertwiner $j_u := j = Fc$, corresponding to $R_u = \sum \psi_i \otimes j\psi_i$, for $r = 1$, we can form its tensor power $j_{u^r}(\xi_1 \dots \xi_r) := j(\xi_r) \dots j(\xi_1)$. With this choice we have

$$R_{u^r} = 1_{u^{r-1}} \otimes R_u \otimes 1_{u^{r-1}} \circ R_{u^{r-1}}.$$

By tensoriality of the inclusion of the full tensor C^*–subcategory of $\mathrm{Rep}(A_o(F))$ in $\mathcal{T}_M$, that with denote by ρ,

$$\rho(R_{u^r}) = 1_{M^{\otimes r-1}} \otimes \rho(R_u) \otimes 1_{M^{\otimes r-1}} \circ \rho(R_{u^{r-1}}) =$$

$$\sum_{i_1, \dots i_r} u_{i_1} \otimes \dots u_{i_r} \otimes {u_{i_r}}^* \otimes \dots {u_{i_1}}^* \in M^{\otimes 2r}.$$

On the other hand, since u^r is real, it is selfconjugate, hence $\overline{u^r} = u^r$. Therefore, for $T \in (\iota, M^{\otimes r})$,

$$(\overline{T} \otimes \xi_1 \dots \xi_r)^* = \overline{1_{M^{\otimes r}} \otimes T^* \circ \rho(R_{u^r})} \otimes j\xi_r \dots j\xi_1 =$$

$$\overline{u_{i_1} \otimes \dots \otimes u_{i_r} < T, u_{i_r}^* \otimes \dots \otimes u_{i_1}^* >} \otimes j\xi_r \dots j\xi_1.$$

We thus need to compute $\sum u_{i_1} \otimes \dots \otimes u_{i_r} < T, u_{i_r}^* \otimes \dots \otimes u_{i_1}^* >$ for $T \in (\iota, M^{\otimes r})$ under the identification $(\iota, M^{\otimes r}) \simeq N' \cap M_{r-1}$. Now it is easy to see, using the trace norm of N, that $E(SX) = E(XS)$ for $S \in N' \cap M$ and $X \in M$, hence, for $r = 1$, and $T \in N' \cap M$,

$$\sum_i u_i < T, u_i^* >= \sum_i u_i E(T^* u_i^*) = \sum_i u_i E({u_i}^* T^*) = T^*.$$

In the general case, if $T \in N' \cap M_{r-1}$, identifying $M^{\otimes r}$ and M_{r-1} as right Hilbert N–modules, $u_{i_1} \otimes \dots \otimes u_{i_r}$ corresponds to a Pimsner–Popa basis for the inclusion $N \subset M_{r-1}$, and the above argument for $r = 1$ shows that $\sum u_{i_1} \otimes \dots \otimes u_{i_r} < T, u_{i_r}^* \otimes \dots \otimes u_{i_1}^* >$ corresponds to the adjoint T^* of T in $N' \cap M_{r-1}$. Summarizing:

5.4 Proposition *Under the restriction of the* N*–bimodule unitary* $\Gamma_r : M^{\otimes r} \to M_{r-1}$*, the antilinear map*

$$J_r : T \in (\iota, M^{\otimes r}) \to 1_{\overline{\rho_{u^{\otimes r}}}} \otimes T^* \circ \rho(R_{u^{\otimes r}}) \in (\iota, M^{\otimes r}) \tag{5.7}$$

corresponds to the antiunitary *–involution

$$T \in N' \cap M_{r-1} \to T^* \in N' \cap M_{r-1}. \tag{5.8}$$

Therefore for the resulting ergodic C^*–action of $A_o(F)$ we have:

$$m(u^r) = dim(N' \cap M_{r-1}), \tag{5.9}$$

where m is the quantum multiplicity.

Hence formula b) in Theorem 4.3 has been established. Before exploiting the operator product (5.5) in $\mathcal{C}$, we complete the proof of Theorem 4.1.

Proof of Theorem 4.1 c)

We need to show that the unique invariant state h is a trace. In the special case that we are considering, $A_o(I_p)$ has an involutive coinverse [31], hence j_w can be chosen antiunitary for all representations w. In particular the solution R_{u^r}, $\overline{R}_{u^r}$ of the conjugate equations is standard. By Cor. A.10 of [28], the solution $\hat{R}_{u^r}$, $\hat{\overline{R}}_{u^r}$ is also standard. We close $\mathcal{T}_M$ under subobjects and then extend the inclusion functor ρ to a relaxed tensor *–functor from $\mathrm{Rep}(A_o(F))$ to this closure, so that ρ_v is now defined for any irreducible subrepresentation v of some u^r. It follows that $J_r \restriction_{(\iota,\rho_v)}$ coincides with an antilinear invertible J_v constructed from a normalized solution of the conjugate equations for v. Since J_r is antiunitary, J_v is antiunitary as well. Let $\hat{\rho}$ be a complete set of irreducible representations of $A_o(F)$. If $v, w \in \hat{\rho}$, $a = \overline{S} \otimes \psi$, $b = \overline{T} \otimes \phi$, $\psi \in H_v$, $\phi \in H_w$, $S \in (\iota, \rho_v)$, $T \in (\iota, \rho_w)$ then $v \otimes w$ contains the trivial representation ι if and only if $w = \overline{v}$, and the multiplicity of ι is 1. Hence $S = \|\overline{R_v}\|^{-1}\overline{R_v} \in (\iota, v \otimes \overline{v})$ is an isometry. It follows that

$$h(ab) = \delta_{w,\overline{v}}\|\overline{R_v}\|^{-2}\overline{\rho(\overline{R_v}^*)S \otimes T} \otimes \overline{R_v}^* \psi \otimes \phi =$$

$$\delta_{w,\overline{v}}\|\overline{R_v}\|^{-2} < J_v S, T >< j_v\psi, \phi > .$$

Similarly,

$$h(ba) = \delta_{v,\overline{w}}\|\overline{R_w}\|^{-2} < J_w T, S >< j_w\phi, \psi >$$

which in turn equals

$$\delta_{w,\overline{v}}\|R_v\|^{-2} < J_v^{-1}T, S >< j_v^{-1}\phi, \psi >=$$

$$\delta_{w,\overline{v}}\|R_v\|^{-2} < J_v S, T >< j_v\psi, \phi >= h(ab).$$

We next spell out the tensor product operation between arrows in Ocneanu's category in terms of the higher relative commutants.

In detail, the isomorphism Γ_r allows us to write down the tensor product $\xi \otimes \eta$ of elements $\xi \in M^{\otimes r}$ and $\eta \in M^{\otimes s}$ in terms of a bilinear map

$$M_{r-1} \times M_{s-1} \to M_{r+s-1}.$$

If, e.g., $\xi = \xi_1 \otimes \xi_2$, $\eta = \eta_1 \otimes \eta_2 \otimes \eta_3$,

$$\Gamma_5(\xi \otimes \eta) = \lambda^{10}\xi_1 e_1 \xi_2 e_2 e_1 \eta_1 e_3 e_2 e_1 \eta_2 e_4 e_3 e_2 e_1 \eta_3 =$$

$$\lambda^9 \Gamma_2(\xi) e_2 e_1 \eta_1 e_3 e_2 e_1 \eta_2 e_4 e_3 e_2 e_1 \eta_3 =$$

$$\lambda^9 \Gamma_2(\xi) e_2 e_1 e_3 e_2 e_4 e_3 (\eta_1 e_1 \eta_2 e_2 e_1 \eta_3) =$$

$$\lambda^6 \Gamma_2(\xi) e_2 e_1 e_3 e_2 e_4 e_3 \Gamma_3(\eta).$$

In general, if $\xi \in M^{\otimes r}$, $\eta \in M^{\otimes s}$ then

$$\Gamma_{r+s}(\xi \otimes \eta) = \lambda^{<r+s>-<r>-<s>} \Gamma_r(\xi)(e_r \dots e_1)(e_{r+1} \dots e_2) \dots (e_{r+s-1} \dots e_s)\Gamma_s(\eta) =$$

$$\lambda^{rs} \Gamma_r(\xi)(e_r \dots e_1)(e_{r+1} \dots e_2) \dots (e_{r+s-1} \dots e_s)\Gamma_s(\eta) = \Gamma_r(\xi) p_{r,s} \Gamma_s(\eta),$$

with $p_{r,s} = p^{(0)}_{r,s}$.

5.5 Proposition *Under the N–bimodule unitary $\Gamma_r : M^{\otimes r} \to M_{r-1}$ the tensor product*

$$\xi \in (\iota, M^{\otimes r}), \eta \in (\iota, M^{\otimes s}) \to \xi \otimes \eta \in (\iota, M^{\otimes r+s}) \tag{5.10}$$

defined in $\mathcal{T}_M$ corresponds to the map

$$S \in N' \cap M_{r-1}, T \in N' \cap M_{s-1} \to S p_{r,s} T \in N' \cap M_{r+s-1}. \tag{5.11}$$

We are left to account for the $\mathrm{Rep}(A_o(F))$–bimodule structure (5.3) used to define $\mathcal{C}_\rho$. Now the full tensor *–subcategory of $\mathrm{Rep}(A_o(F))$ generated by u is generated, as a linear category, by the arrows $1_{u^{\otimes r}} \otimes R_u \otimes 1_{u^{\otimes s}}$, $r, s \geq 0$, and by their adjoints. Therefore we are led to compute the composition of arrows of the form $1_{M^{\otimes r}} \otimes R \otimes 1_{M^{\otimes s}} \circ T$ and $1_{M^{\otimes r}} \otimes R^* \otimes 1_{M^{\otimes s}} \circ S$ with $T \in (\iota, M^{\otimes r+s})$, $S \in (\iota, M^{\otimes r+s+2})$, $r, s \geq 0$, in terms of the higher relative commutants. For $\xi \in M^{\otimes r}$, $\eta \in M^{\otimes s}$,

$$1_{M^{\otimes r}} \otimes R \otimes 1_{M^{\otimes s}} \xi \otimes \eta = \xi \otimes R \otimes \eta,$$

hence

$$\Gamma_{r+2+s}(1_{M^{\otimes r}} \otimes R \otimes 1_{M^{\otimes s}}(\xi \otimes \eta)) =$$

$$\Gamma_{r+2}(\xi \otimes R) p_{r+2,s} \Gamma_s(\eta) =$$

$$\Gamma_r(\xi) p_{r,2} \Gamma_2(R) p_{r+2,s} \Gamma_s(\eta).$$

Now

$$\Gamma_2(R) = \sum_i \Gamma_2(u_i \otimes u_i^*) = \lambda \sum_i u_i e_1 u_i^* = \lambda.$$

Therefore

$$\Gamma_{r+2+s}(1_{M^{\otimes r}} \otimes R \otimes 1_{M^{\otimes s}}(\xi \otimes \eta)) = \lambda \Gamma_r(\xi) p_{r,2} p_{r+2,s} \Gamma_s(\eta). \tag{5.12}$$

We therefore need to write $p_{r,2}p_{r+2,s}$ as a reduced word in the algebra generated by the Jones projections $e_1, \ldots, e_{r+s+1}$. It is known that elements of the form

$$(e_{j_1}e_{j_1-1}\ldots e_{i_1})(e_{j_2}\ldots e_{j_2-1}\ldots e_{i_2})\ldots(e_{j_p}e_{j_p-1}\ldots e_{i_p})$$

are in reduced form if $j_1 < j_2 < \cdots < j_p$ and $i_1 < i_2 < \cdots < i_p$ [10]. The following lemma is useful.

5.6 Lemma *For $p \leq j \leq r < s$ we have*

$$(e_re_{r-1}\ldots e_j)(e_se_{s-1}\ldots e_p) = \lambda^{-2}(e_r\ldots e_p)(e_s\ldots e_{j+2}) \quad \text{for } s > j+1,$$

$$(e_re_{r-1}\ldots e_j)(e_se_{s-1}\ldots e_p) = \lambda^{-2}(e_r\ldots e_p) \quad \text{for } s = j+1.$$

Proof We do the computation only in the case $s > j+1$.

$$(e_re_{r-1}\ldots e_j)(e_s\ldots e_{j+2}e_{j+1}e_j\ldots e_p) = (e_re_{r-1}\ldots e_{j+1})(e_s\ldots e_{j+2}e_je_{j+1}e_j\ldots e_p) =$$

$$\lambda^{-2}(e_re_{r-1}\ldots e_{j+1})(e_s\ldots e_{j+2}e_j\ldots e_p) = \lambda^{-2}(e_re_{r-1}\ldots e_{j+1}e_j\ldots e_p)(e_s\ldots e_{j+2}).$$

5.7 Lemma *For $s \geq 0$ and $r \geq s$ we have:*

$$p_{r,2}p_{r+2,s} = p_{r,s}p^{(2s)}_{r-s,2}.$$

Proof The formula is obvious for $s = 0$. We can then assume $s > 0$. We write down the left hand side explicitly:

$$p_{r,2}p_{r+2,s} = \lambda^{2r+2s+rs}(e_r\ldots e_1)(e_{r+1}\ldots e_2)(e_{r+2}\ldots e_1)\ldots(e_{r+s+1}\ldots e_s).$$

We have $2+s$ factors between parentheses. Let us apply the previous lemma iteratively between the second and the third factor. If $r+2 > 3$, i.e. $r > 1$, we have,

$$(e_{r+1}\ldots e_2)(e_{r+2}\ldots e_1) = \lambda^{-2}(e_{r+1}\ldots e_1)(e_{r+2}\ldots e_{2+2}).$$

We proceed to apply the lemma to the new third and old fourth factors: if $r+3 > 5$, i.e. $r > 2$,

$$(e_{r+2}\ldots e_{2+2})(e_{r+3}\ldots e_2) = \lambda^{-2}(e_{r+2}\ldots e_2)(e_{r+3}\ldots e_{2+4}).$$

If $n > s$ after s iterations of the lemma we still find a product of $2+s$ factors:

$$p_{r,2}p_{r+2,s} = \lambda^{2r+rs}(e_r\ldots e_1)(e_{r+1}\ldots e_1)\ldots(e_{r+s}\ldots e_s)(e_{r+s+1}\ldots e_{2+2s}).$$

If $r = s$ instead, the computation goes through but the last application of the lemma requires the second formula for the reduced word. Hence the last factor needs to be replaced by the identity:

$$p_{r,2}p_{r+2,r} = \lambda^{2r+r^2}(e_r\ldots e_1)(e_{r+1}\ldots e_1)(e_{r+2}\ldots e_2)\ldots(e_{2r}\ldots e_r).$$

If $r > s$ we apply the lemma iteratively, but now only s times (in spite of the $s+2$ factors) to the first two factors, the second and third factor and so on. We get

$$p_{r,2}p_{r+2,s} = \lambda^{2r+rs-2s}(e_r \dots e_1)\dots(e_{r+s-1}\dots e_s)(e_{r+s}\dots e_{2s+1})(e_{r+s+1}\dots e_{2+2s}) =$$

$$\lambda^{2(r-s)}p_{r,s}(e_{r+s}\dots e_{1+2s})(e_{r+s+1}\dots e_{2+2s}) = p_{r,s}p^{(2s)}_{r-s,2},$$

and the formula is proved. Now assume $r = s$. Then the right hand side of the desired formula reduces to $p_{r,r}$. The same computation goes through except at the last s-th iteration, where

$$p_{r,2}p_{r+2,r} = \lambda^{r^2}(e_r \dots e_1)\dots(e_{2r-1}\dots e_r) = p_{r,r}.$$

The proof is now complete. It is now not difficult to interpret left tensoring by the translates of R in terms of the Jones tower.

5.8 Proposition *For $r, s \geq 0$ and $\zeta \in M^{\otimes r+s}$, we have:*

a) *for $r > s$, $\Gamma_{r+2+s}(1_{M^{\otimes r}} \otimes R \otimes 1_{M^{\otimes s}} \circ \zeta) = \lambda\Gamma_{r+s}(\zeta)p^{(2s)}_{r-s,2}$,*

b) *for $r = s$, $\Gamma_{2r+2}(1_{M^{\otimes r}} \otimes R \otimes 1_{M^{\otimes r}} \circ \zeta) = \lambda\Gamma_{2r}(\zeta)$,*

c) *for $r < s$, $\Gamma_{r+2+s}(1_{M^{\otimes r}} \otimes R \otimes 1_{M^{\otimes s}} \circ \zeta) = \lambda p^{(2r)}_{2,s-r}\Gamma_{r+s}(\zeta)$.*

Proof For $r = 0$ or $s = 0$ the formula can be deduced from $\Gamma_2(R) = \lambda$ and the computation of the tensor product given in Prop. 5.5. We can then assume $r, s > 0$. In this case a) and b) follow from the previous lemma, property (5.12) and the fact that the e_k commute with M_{s-1} for $k \geq 2s+1$ (recall (3.1)) We prove c). Let us choose ζ of the form $\zeta = \xi \otimes \eta_1 \otimes \eta_2$ with $\xi, \eta_1 \in M^{\otimes r}$, $\eta_2 \in M^{\otimes k}$, where $s = r + k$. Then

$$\Gamma_{r+2+s}(1_{M^{\otimes r}} \otimes R \otimes 1_{M^{\otimes s}} \circ \zeta) = \Gamma_{r+s+2}(\xi \otimes R \otimes \eta_1 \otimes \eta_2) =$$

$$\Gamma_{2r+2}(\xi \otimes R \otimes \eta_1)p_{2r+2,k}\Gamma_k(\eta_2) = \lambda\Gamma_{2r}(\xi \otimes \eta_1)p_{2r+2,k}\Gamma_k(\eta_2).$$

On the other hand

$$p_{2r+2,k} = \lambda^{k(2r+2)}((e_{2r+2}e_{2r+1})e_{2r}\dots e_1)\dots((e_{2r+k+1}e_{2r+k})e_{2r+k-1}\dots e_k) =$$

$$\lambda^{2k}(e_{2r+2}e_{2r+1})(e_{2r+3}e_{2r+2})\dots(e_{r+s+1}e_{r+s})p_{2r,k},$$

which implies

$$\Gamma_{r+2+s}(1_{M^{\otimes r}} \otimes R \otimes 1_{M^{\otimes s}} \circ \zeta) =$$

$$\lambda^{1+2(s-r)}\Gamma_{2r}(\xi \otimes \eta_1)(e_{2r+2}e_{2r+1})(e_{2r+3}e_{2r+2})\dots(e_{r+s+1}e_{r+s})p_{2r,k}\Gamma_k(\eta_2) =$$

$$\lambda^{1+2(s-r)}(e_{2r+2}e_{2r+1})(e_{2r+3}e_{2r+2})\dots(e_{r+s+1}e_{r+s})\Gamma_{r+s}(\zeta) = \lambda p^{(2r)}_{2,s-r}.$$

We next compute the operators in the Jones tower corresponding to tensoring on the left by $1_{M^{\otimes r}} \otimes R^* \otimes 1_{M^{\otimes s}}$.

5.9 Proposition *For $r, s \geq 0$, $\zeta \in M^{\otimes r+s+2}$ we have:*

a) *for* $r > s$, $\Gamma_{r+s}(1_{M^{\otimes r}} \otimes R^* \otimes 1_{M^{\otimes s}} \circ \zeta) = \lambda E_{r+s} E_{r+s+1}(\Gamma_{r+s+2}(\zeta)(p^{(2s)}_{r-s,2})^*)$,

b) *for* $r = s$, $\Gamma_{2r}(1_{M^{\otimes r}} \otimes R^* \otimes 1_{M^{\otimes r}} \circ \zeta) = \lambda E_{r+s} E_{r+s+1}(\Gamma_{2r+2}(\zeta))$,

c) *for* $r < s$, $\Gamma_{r+s}(1_{M^{\otimes r}} \otimes R^* \otimes 1_{M^{\otimes s}} \circ \zeta) = \lambda E_{r+s} E_{r+s+1}((p^{(2r)}_{2,s-r})^* \Gamma_{r+s+2}(\zeta))$.

Proof With respect to the inner products the N–bimodule operators $1_{M^{\otimes r}} \otimes R \otimes 1_{M^{\otimes s}} : M^{\otimes r+s} \to M^{\otimes r+s+2}$ and $1_{M^{\otimes r}} \otimes R^* \otimes 1_{M^{\otimes s}} : M^{\otimes r+s+2} \to M^{\otimes r+s}$ are adjoints of each other Recall that, by Prop. 10.5, $\Gamma_p : M^{\otimes p} \to M_{p-1}$ is a N–bimodule unitary if M_{p-1} is regarded as a N–bimodule with inner product defined by the conditional expectation $E_{(p)} = EE_1 \dots E_{p-1} : M_{p-1} \to N$. Therefore we need to compute

$$\Gamma_{r+s} 1_{M^{\otimes r}} \otimes R^* \otimes 1_{M^{\otimes s}} \Gamma^*_{r+s+2} = (\Gamma_{r+s+2} 1_{M^{\otimes r}} \otimes R \Gamma^*_{r+s})^*.$$

Now by the previous lemma, for $r > s$, $(\Gamma_{r+s+2} 1_{M^{\otimes r}} \otimes R\Gamma^*_{r+s}) : M_{r+s-1} \to M_{r+s+1}$ is right multiplication by the element $A = \lambda p^{2s}_{r-s,2} \in M_{r+s+1}$. Hence its adjoint, r^*_A, is

$$< X, r^*_A Y > = < XA, Y > = E_{(r+s+2)}(A^* X^* Y) = E_{(r+s+2)}(X^* Y A^*)$$

where $X \in M_{r+s-1}$, $Y \in M_{r+s+1}$, as A commutes with N. Hence

$$< X, r^*_A Y > = E_{(r+s)}(X^* E_{r+s} E_{r+s+1}(Y A^*)),$$

as $X \in M_{r+s-1}$, and this shows that $r^*_A Y = E_{r+s} E_{r+s+1}(Y A^*)$. The remaining cases follow similarly.

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