arXiv:0805.2136v2 [math.SP] 15 Jan 2009

SPECTRAL GAPS OF THE ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH SINGULAR PERIODIC POTENTIALS

VLADIMIR MIKHAILETS AND VOLODYMYR MOLYBOGA

In the memory of A.Ya. Povzner.

Abstract. The behaviour of the lengths of spectral gaps $\{\gamma_n(q)\}_{n=1}^{\infty}$ of the Hill-Schrödinger operators

$$S(q)u = -u'' + q(x)u, \quad u \in \mathrm{Dom}\,(S(q))$$

with real-valued 1-periodic distributional potentials $q(x) \in H^{-1}_{1\text{-}per}(\mathbb{R})$ is studied. We show that they exhibit the same behaviour as the Fourier coefficients $\{\widehat{q}(n)\}_{n=-\infty}^{\infty}$ of the potentials $q(x)$ with respect to the weighted sequence spaces $h^{s,\varphi}$, $s > -1$, $\varphi \in \mathrm{SV}$. The case $q(x) \in L^2_{1\text{-}per}(\mathbb{R})$, $s \in \mathbb{Z}_+$, $\varphi \equiv 1$ corresponds to the Marchenko-Ostrovskii Theorem.

1. Introduction

The Hill-Schrödinger operators

$$S(q)u := -u'' + q(x)u, \quad u \in \mathrm{Dom}\,(S(q))$$

with real-valued 1-periodic distributional potentials $q(x) \in H^{-1}_{1\text{-}per}(\mathbb{R})$ are well defined on the Hilbert space $L^2(\mathbb{R})$ in the following *equivalent* basic ways [MiMl6]:

- as form-sum operators;
- as quasi-differential operators;
- as limits of operators with smooth 1-periodic potentials in the norm resolvent sense.

The operators $S(q)$ are lower semibounded and self-adjoint on the Hilbert space $L^2(\mathbb{R})$. Their spectra are absolutely continuous and have a band and gap structure as in the classical case of $L^2_{1\text{-}per}(\mathbb{R})$-potentials [HrMk, Krt, DjMt3, MiMl6].

The object of our investigation is the behaviour of the lengths of spectral gaps. Under the assumption

$$q(x) = \sum_{k\in\mathbb{Z}} \widehat{q}(k) e^{ik2\pi x} \in H^{-1+}_{1\text{-}per}(\mathbb{R},\mathbb{R}), \tag{1}$$

that is

$$\sum_{k\in\mathbb{Z}} (1+2|k|)^{2s} |\widehat{q}(k)|^2 < \infty \quad \forall s > -1, \quad \text{and} \quad \mathrm{Im}\, q(x) = 0,$$

we will prove many terms asymptotic estimates for the lengths $\{\gamma_n(q)\}_{n=1}^{\infty}$ and midpoints $\{\tau_n(q)\}_{n=1}^{\infty}$ of spectral gaps of the Hill-Schrödinger operators $S(q)$ (Theorem 1). These estimates enable us to establish relationship between the rate of *decreasing/increasing* of the lengths of spectral gaps and the *regularity* of the singular potentials (Theorem 2 and Theorem 3).

It is well known that if the potentials

$$q(x) = \sum_{k\in\mathbb{Z}} \widehat{q}(k) e^{ik2\pi x} \in L^2_{1\text{-}per}(\mathbb{R},\mathbb{R}),$$

2000 *Mathematics Subject Classification.* Primary 34L40; Secondary 47A10, 47A75.
Key words and phrases. Hill-Schrödinger operators, singular potentials, spectral gaps.

i.e., when

$$\sum_{k\in\mathbb{Z}} |\widehat{q}(k)|^2 < \infty, \quad \text{and} \quad \operatorname{Im} q(x) = 0,$$

then the Hill-Schrödinger operators $S(q)$ are lower semibounded and self-adjoint operators on the Hilbert space $L^2(\mathbb{R})$ with absolutely continuous spectra which have a zone structure [DnSch2, ReSi4].

Spectra $\operatorname{spec}(S(q))$ are defined by the location of the endpoints $\{\lambda_0(q), \lambda_n^{\pm}(q)\}_{n=1}^{\infty}$ of spectral gaps which satisfy the following inequalities:

$$-\infty < \lambda_0(q) < \lambda_1^-(q) \le \lambda_1^+(q) < \lambda_2^-(q) \le \lambda_2^+(q) < \cdots .$$

Moreover, for even/odd numbers $n \in \mathbb{Z}_+$ the endpoints of spectral gaps are eigenvalues of the periodic/semiperiodic problems on the interval $[0,1]$:

$$S_{\pm}(q)u := -u'' + q(x)u = \lambda u,$$
$$\operatorname{Dom}(S_{\pm}(q)) := \left\{ u \in H^2[0,1] \,\middle|\, u^{(j)}(0) = \pm\, u^{(j)}(1),\ j = 0,1 \right\} \equiv H^2_{\pm}[0,1].$$

Spectral bands (stability or tied zones),

$$\mathcal{B}_0(q) := [\lambda_0(q), \lambda_1^-(q)], \qquad \mathcal{B}_n(q) := [\lambda_n^+(q), \lambda_{n+1}^-(q)], \quad n \in \mathbb{N},$$

are characterized as a locus of those real $\lambda \in \mathbb{R}$ for which all solutions of the equation $S(q)u = \lambda u$ are bounded. On the other hand, spectral gaps (instability or forbidden zones),

$$\mathcal{G}_0(q) := (-\infty, \lambda_0(q)), \qquad \mathcal{G}_n(q) := (\lambda_n^-(q), \lambda_n^+(q)), \quad n \in \mathbb{N},$$

are a locus of those real $\lambda \in \mathbb{R}$ for which any nontrivial solution of the equation $S(q)u = \lambda u$ is unbounded.

Due to Marchenko and Ostrovskii [MrOs] the endpoints of spectral gaps of the Hill-Schrödinger operators $S(q)$ satisfy the asymptotic estimates

$$\lambda_n^{\pm}(q) = n^2\pi^2 + \widehat{q}(0) \pm |\widehat{q}(n)| + h^1(n), \quad n \to \infty. \tag{2}$$

As a consequence, for the lengths of spectral gaps,

$$\gamma_n(q) := \lambda_n^+ - \lambda_n^-, \quad n \in \mathbb{N},$$

the following asymptotic formulae are fulfilled,

$$\gamma_n(q) = 2\,|\widehat{q}(n)| + h^1(n), \quad n \to \infty. \tag{3}$$

Hochstadt [Hchs] ($\Rightarrow$) and Marchenko, Ostrovskii [MrOs], McKean, Trubowitz [McKTr] ($\Leftarrow$) proved that the potential $q(x)$ is an infinitely differentiable function if and only if the lengths of spectral gaps $\{\gamma_n(q)\}_{n=1}^{\infty}$ decrease faster than arbitrary power of $1/n$:

$$q(x) \in C^{\infty}_{1\text{-}per}(\mathbb{R},\mathbb{R}) \Leftrightarrow \gamma_n(q) = O(n^{-k}),\ n \to \infty \quad \forall k \in \mathbb{Z}_+.$$

Marchenko and Ostrovskii [MrOs] discovered that

$$q(x) \in H^k_{1\text{-}per}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^k, \quad k \in \mathbb{Z}_+. \tag{4}$$

The relationship (4) was extended by Kappeler, Mityagin [KpMt] ($\Rightarrow$) and Djakov, Mityagin [DjMt1] ($\Leftarrow$) (see also the survey [DjMt2] and the references therein) on the case of symmetric, monotonic, submultiplicative and strictly subexponential weights $\Omega = \{\Omega(n)\}_{n\in\mathbb{Z}}$:

$$q(x) \in H^{\Omega}_{1\text{-}per}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{\Omega}.$$

Pöschel [Psch] proved the latter statement in a quite different way.

Here and throughout the remainder of the paper we use the complex Hilbert spaces $H^w_{1\text{-}per}(\mathbb{R})$ (as well as $H^w_{\pm}[0,1]$) of 1-periodic functions and distributions defined by means of their Fourier coefficients:

$$f(x)=\sum_{k\in\mathbb{Z}}\widehat{f}(k)e^{ik2\pi x}\in H^w_{1\text{-}per}(\mathbb{R})\Leftrightarrow\left\{\widehat{f}(k)\right\}_{k\in\mathbb{Z}}\in h^w,$$

$$h^w=\left\{a=\{a(k)\}_{k\in\mathbb{Z}}\;\middle|\;\|a\|_{h^w}=\left(\sum_{k\in\mathbb{Z}}w^2(k)|a(k)|^2\right)^{1/2}<\infty\right\}.$$

Basically we use the power weights

$$w_s=\{w_s(k)\}_{k\in\mathbb{Z}}:\qquad w_s(k):=(1+2|k|)^s,\quad s\in\mathbb{R}.$$

The corresponding spaces we denote as

$$H^{w_s}_{1\text{-}per}(\mathbb{R})\equiv H^s_{1\text{-}per}(\mathbb{R}),\ H^{w_s}_{\pm}[0,1]\equiv H^s_{\pm}[0,1],\quad\text{and}\quad h^{w_s}\equiv h^s,\ s\in\mathbb{R}.$$

For more details, see Appendix.

2. Main results

As we already remarked, under the assumption (1) the Hill-Schrödinger operators $S(q)$ are lower semibounded and self-adjoint on the Hilbert space $L^2(\mathbb{R})$. Their spectra are absolutely continuous and have a classical zone structure [HrMk, Krt, DjMt3, MiMl6, MiSb].

Using the results of the papers [KpMh, Mhr], the Isospectral Theorem 5 and [MiMl6, Theorem C] we prove uniform many terms asymptotic estimates for the lengths of spectral gaps $\{\gamma_n(q)\}_{n=1}^{\infty}$ and their midpoints $\{\tau_n(q)\}_{n=1}^{\infty}$,

$$\tau_n(q):=\frac{\lambda_n^+(q)+\lambda_n^-(q)}{2},\quad n\in\mathbb{N}.$$

Theorem 1 ([MiMl3, Mlb2]). *Let $q(x)\in H^{-\alpha}_{1\text{-}per}(\mathbb{R},\mathbb{R})$, $\alpha\in[0,1)$. Then for any $\varepsilon>0$ uniformly on the bounded sets of distributions $q(x)$ in the corresponding Sobolev spaces $H^{-\alpha}_{1\text{-}per}(\mathbb{R})$ the lengths $\{\gamma_n(q)\}_{n=1}^{\infty}$ and midpoints $\{\tau_n(q)\}_{n=1}^{\infty}$ of spectral gaps of the Hill-Schrödinger operators $S(q)$ for $n\geq n_0\left(\|q\|_{H^{-\alpha}_{1\text{-}per}(\mathbb{R})}\right)$ satisfy the following asymptotic formulae:*

$$\gamma_n(q)=2\,|\widehat{q}(n)|+h^{1-2\alpha-\varepsilon}(n),\tag{5}$$

$$\tau_n(q)=n^2\pi^2+\widehat{q}(0)+h^{1-2\alpha-\varepsilon}(n).\tag{6}$$

Corollary ([MiMl3, Mlb2]). *Let $q(x)\in H^{-\alpha}_{1\text{-}per}(\mathbb{R},\mathbb{R})$ with $\alpha\in[0,1)$. Then for any $\varepsilon>0$ uniformly by $q(x)$ for the endpoints of spectral gaps of the Hill-Schrödinger operators $S(q)$ the following asymptotic estimates are fulfilled:*

$$\lambda_n^{\pm}(q)=n^2\pi^2+\widehat{q}(0)\pm|\widehat{q}(n)|+h^{1-2\alpha-\varepsilon}(n).$$

Now, we can describe a bilateral relationship between the rate of decreasing/increasing of the lengths of spectral gaps $\{\gamma_n(q)\}_{n=1}^{\infty}$ and the regularity of the potentials $q(x)$ in the refined scale.

Let

$$w_{s,\varphi}=\{w_{s,\varphi}(k)\}_{k\in\mathbb{Z}}:\qquad w_{s,\varphi}(k):=(1+2|k|)^s\,\varphi(|k|),\quad s\in\mathbb{R},\ \varphi\in\mathrm{SV},$$

where φ is a slowly varying on $+\infty$ in a sense of Karamata function [Snt]. It means that it is a positive, measurable on $[A,\infty)$, $A>0$ function obeying the condition

$$\lim_{t\to+\infty}\frac{\varphi(\lambda t)}{\varphi(t)}=1,\quad\lambda>0.$$

For example,

$$\varphi(t) = (\log t)^{r_1}(\log\log t)^{r_2}\dots(\log\dots\log t)^{r_k} \in \mathrm{SV}, \quad \{r_1,\dots,r_k\} \subset \mathbb{R},\ k \in \mathbb{N}.$$

The Hörmander spaces

$$H^{w_{s,\varphi}}_{1\text{-}per}(\mathbb{R}) \equiv H^{s,\varphi}_{1\text{-}per}(\mathbb{R}) \simeq H^{s,\varphi}(\mathbb{S}), \quad \mathbb{S} := \mathbb{R}/2\pi\mathbb{Z},$$

and the weighted sequence spaces

$$h^{w_{s,\varphi}} \equiv h^{s,\varphi}$$

form the refined scales:

$$H^{s+\varepsilon}_{1\text{-}per}(\mathbb{R}) \hookrightarrow H^{s,\varphi}_{1\text{-}per}(\mathbb{R}) \hookrightarrow H^{s-\varepsilon}_{1\text{-}per}(\mathbb{R}), \tag{7}$$

$$h^{s+\varepsilon} \hookrightarrow h^{s,\varphi} \hookrightarrow h^{s-\varepsilon}, \qquad s \in \mathbb{R},\ \varepsilon > 0,\ \varphi \in \mathrm{SV}, \tag{8}$$

which in a general situation were studied by Mikhailets and Murach [MiMr].

The following statements show that the sequence $\{\gamma_n(q)\}_{n=1}^{\infty}$ has the same behaviour as the Fourier coefficients $\{\widehat{q}(n)\}_{n=-\infty}^{\infty}$ with respect to the refined scale $\{h^{s,\varphi}\}_{s\in\mathbb{R},\varphi\in\mathrm{SV}}$.

Theorem 2. *Let $q(x) \in H^{-1+}_{1\text{-}per}(\mathbb{R},\mathbb{R})$. Then*

$$q(x) \in H^{s,\varphi}_{1\text{-}per}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s,\varphi}, \quad s \in (-1,0], \varphi \in \mathrm{SV}.$$

Note that the Hörmander spaces $H^{s,\varphi}_{1\text{-}per}(\mathbb{R})$ with $\varphi \equiv 1$ coincide with the Sobolev spaces,

$$H^{s,1}_{1\text{-}per}(\mathbb{R}) \equiv H^{s}_{1\text{-}per}(\mathbb{R}), \quad \text{and} \quad h^{s,1} \equiv h^{s}, \quad s \in \mathbb{R}.$$

Corollary ([MiMl3, Mlb2])**.** *Let $q(x) \in H^{-1+}_{1\text{-}per}(\mathbb{R},\mathbb{R})$, then*

$$q(x) \in H^{s}_{1\text{-}per}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s}, \quad s \in (-1,0]. \tag{9}$$

Theorem 2 together with [KpMt, Theorem 1.2], and the properties (7) and (8), involve the following extension of the Marchenko-Ostrovskii Theorem (4).

Theorem 3. *Let $q(x) \in H^{-1+}_{1\text{-}per}(\mathbb{R},\mathbb{R})$. Then*

$$q(x) \in H^{s,\varphi}_{1\text{-}per}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s,\varphi}, \quad s \in (-1,\infty), \varphi \in \mathrm{SV}.$$

In particular,

$$q(x) \in H^{s}_{1\text{-}per}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s}, \quad s \in (-1,\infty).$$

Remark. In the preprint [DjMt3] the authors announced without a proof the more general statement:

$$q(x) \in H^{\widehat{\Omega}}_{1\text{-}per}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{\widehat{\Omega}}, \quad \widehat{\Omega} = \left\{\frac{\Omega(n)}{1+2|n|}\right\}_{n\in\mathbb{Z}},$$

where the weights $\Omega = \{\Omega(n)\}_{n\in\mathbb{Z}}$ are supposed to be symmetric, monotonic, submultiplicative and strictly subexponential ones. This result contains the limiting case

$$q(x) \in H^{-1}_{1\text{-}per}(\mathbb{R},\mathbb{R}) \setminus H^{-1+}_{1\text{-}per}(\mathbb{R},\mathbb{R}).$$

An implication

$$q(x) \in H^{-1}_{1\text{-}per}(\mathbb{R},\mathbb{R}) \Rightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{-1}$$

was proved in the paper [Krt].

3. Proofs

Spectra of the Hill-Schrödinger operators $S(q)$, $q(x) \in H^{-1}_{1\text{-}per}(\mathbb{R},\mathbb{R})$ are defined by the endpoints $\{\lambda_0(q), \lambda_n^{\pm}(q)\}_{n=1}^{\infty}$ of spectral gaps. The endpoints as in the case of $L^2_{1\text{-}per}(\mathbb{R})$-potentials satisfy the inequalities:

$$-\infty < \lambda_0(q) < \lambda_1^-(q) \le \lambda_1^+(q) < \lambda_2^-(q) \le \lambda_2^+(q) < \cdots .$$

For even/odd numbers $n \in \mathbb{Z}_+$ they are eigenvalues of the periodic/semiperiodic problems on the interval $[0,1]$ [MiMl6, Theorem C],

$$S_{\pm}(q)u = \lambda u.$$

The operators

$$S_{\pm}u \equiv S_{\pm}(q)u := D_{\pm}^2 u + q(x)u,$$

- $D_{\pm}^2 := -d^2/dx^2$, $\mathrm{Dom}\,(D_{\pm}^2) = H_{\pm}^2[0,1]$;
- $q(x) = \sum\limits_{k\in\mathbb{Z}} \widehat{q}(k)\, e^{i\,k2\pi x} \in H_+^{-1}([0,1],\mathbb{R})$;
- $\mathrm{Dom}\,(S_{\pm}(q)) = \left\{ u \in H_{\pm}^1[0,1] \;\middle|\; D_{\pm}^2 u + q(x)u \in L^2(0,1) \right\}$,

are well defined on the Hilbert space $L^2(0,1)$ as lower semibounded, self-adjoint form-sum operators, and they have the pure discrete spectra

$$\mathrm{spec}\,(S_{\pm}(q)) = \left\{ \lambda_0[S_+(q)],\ \lambda_{2n-1}^{\pm}[S_-(q)],\ \lambda_{2n}^{\pm}[S_+(q)] \right\}_{n=1}^{\infty}.$$

In the papers [MiMl3, Mlb2, MiMl4, MiMl5] the authors meticulously investigated the more general periodic/semiperiodic form-sum operators

$$S_{m,\pm}(V) := D_{\pm}^{2m} \dotplus V(x), \quad V(x) \in H_+^{-m}[0,1],\ m \in \mathbb{N},$$

on the Hilbert space $L^2(0,1)$.

So, we need to find precise asymptotic estimates for the operators $S_{\pm}(q)$ eigenvalues. It is quite difficult problem as the form-sum operators $S_{\pm}(q)$ are not convenient for investigation. We also cannot apply approach developed by Savchuk and Shkalikov (see the survey [SvSh] and the references therein) considering the operators $S_{\pm}(q)$ as quasi-differential ones as the periodic/semiperiodic boundary conditions are not strongly regular by Birkhoff. Therefore we propose an alternative approach which is based on isospectral transformation of the problem.

Kappeler and Möhr [KpMh, Mhr] investigated the second order differential operators $L_{\pm}(q)$, $q(x) \in H_+^{-1}([0,1],\mathbb{R})$ (in general, with complex-valued potentials) defined on the *negative* Sobolev spaces $H_{\pm}^{-1}[0,1]$,

$$L_{\pm} \equiv L_{\pm}(q) := D_{\pm}^2 + q(x), \quad \mathrm{Dom}\,(L_{\pm}(q)) = H_{\pm}^1[0,1].$$

They established that the operators $L_{\pm}(q)$ with $q(x) \in H_+^{-\alpha}([0,1],\mathbb{R})$, $\alpha \in [0,1)$ have the real-valued discrete spectra

$$\mathrm{spec}\,(L_{\pm}(q)) = \left\{ \lambda_0[L_+(q)],\ \lambda_{2n-1}^{\pm}[L_-(q)],\ \lambda_{2n}^{\pm}[L_+(q)] \right\}_{n=1}^{\infty}$$

such that

$$\left|\lambda_n^{\pm}[L_{\pm}(q)] - n^2\pi^2 - \widehat{q}(0)\right| \le Cn^{\alpha}, \quad n \ge n_0\left(\|q\|_{H_+^{-\alpha}[0,1]}\right).$$

More precisely, for the values

$$\gamma_n[L_{\pm}(q)] := \lambda_n^+[L_{\pm}(q)] - \lambda_n^-[L_{\pm}(q)], \quad n \in \mathbb{N},$$

$$\tau_n[L_{\pm}(q)] := \frac{\lambda_n^+[L_{\pm}(q)] + \lambda_n^-[L_{\pm}(q)]}{2}, \quad n \in \mathbb{N}$$

they proved the following result.

Proposition 4 (Kappeler, Möhr [KpMh, Mhr]). *Let $q(x) \in H_+^{-\alpha}([0,1],\mathbb{R})$, and $\alpha \in [0,1)$. Then for any $\varepsilon > 0$ uniformly on the bounded sets of distributions $q(x)$ in the Sobolev spaces $H_+^{-\alpha}[0,1]$ for the operators $L_\pm(q)$ values $\{\gamma_n[L_\pm(q)]\}_{n=1}^\infty$ and $\{\tau_n[L_\pm(q)]\}_{n=1}^\infty$ for $n \geq n_0\left(\|q\|_{H_+^{-\alpha}[0,1]}\right)$ the following asymptotic estimates are fulfilled:*

$$i) \quad \left\{\min_{\pm}\left|\gamma_n[L_\pm(q)] \pm 2\sqrt{(\widehat{q}+\omega)(-n)(\widehat{q}+\omega)(n)}\right|\right\}_{n\in\mathbb{N}} \in h^{1-2\alpha-\varepsilon},$$

$$ii) \quad \tau_n[L_\pm(q)] = n^2\pi^2 + \widehat{q}(0) + h^{1-2\alpha-\varepsilon}(n),$$

where the convolution

$$\{\omega(n)\}_{n\in\mathbb{Z}} \equiv \left\{\frac{1}{\pi^2}\sum_{k\in\mathbb{Z}\setminus\{\pm n\}}\frac{\widehat{q}(n-k)\widehat{q}(n+k)}{n^2-k^2}\right\}_{n\in\mathbb{Z}} \in \begin{cases} h^{1-\alpha}, & \alpha\in[0,1/2), \\ h^{3/2-2\alpha-\delta}, & \alpha\in[1/2,1) \end{cases}$$

with any $\delta > 0$ (see the Convolution Lemma [KpMh, Mhr]*).*

Remark. In the papers [Mlb1, MiMl1, MiMl2, Mlb2] more general operators

$$L_{m,\pm}(V) := D_\pm^{2m} + V(x), \quad V(x) \in H_+^{-m}[0,1],\ m \in \mathbb{N}$$

on the spaces $H_\pm^{-m}[0,1]$ were studied. In particular, the analogue of Proposition 4 was proved.

The following statement is an essential point of our approach.

Theorem 5 (Isospectral Theorem [MiMl3, Mlb2]). *The operators $S_\pm(q)$ and $L_\pm(q)$ are isospectral ones:*

$$\operatorname{spec}(S_\pm(q)) = \operatorname{spec}(L_\pm(q)).$$

Proof. The injections

$$\operatorname{spec}(S_\pm(q)) \subset \operatorname{spec}(L_\pm(q))$$

are obvious since

$$S_\pm(q) \subset L_\pm(q).$$

Let prove the inverse injections

$$\operatorname{spec}(L_\pm(q)) \subset \operatorname{spec}(S_\pm(q)).$$

Let $\lambda \in \operatorname{spec}(L_\pm(q))$, and f be a correspondent eigenvector or rootvector. Therefore

$$(L_\pm(q) - \lambda Id)f = g, \quad f, g \in \operatorname{Dom}(L_\pm(q)) = H_\pm^1[0,1],$$

where f is an eigenfunction if $g = 0$, and a rootvector if $g \neq 0$.

So, we have got

$$L_\pm(q)f = \lambda Idf + g \in H_\pm^1[0,1],$$

i.e.,

$$L_\pm(q)f = D_\pm^2 f + q(x)f \in L^2(0,1).$$

Thus we have proved that $f \in \operatorname{Dom}(S_\pm(q))$. In the case when f is a rootvector ($g \neq 0$) in a similar fashion we show that $g \in \operatorname{Dom}(S_\pm(q))$ also. Continuing this process until necessary (draw attention that it is finite as the eigenvalue λ has a finite algebraic multiplicity) we obtain that all correspondent to λ eigenvectors and rootvectors belong to the operators $S_\pm(q)$ domains $\operatorname{Dom}(S_\pm(q))$. Consequently we can conclude that

$$\lambda \in \operatorname{spec}(S_\pm(q)),$$

and the required injections

$$\operatorname{spec}(L_\pm(q)) \subset \operatorname{spec}(S_\pm(q))$$

have been proved.

The proof is complete. □

Now, Theorem 1 follows from Proposition 4, the Isospectral Theorem 5 and [MiMl6, Theorem C], since

$$\widehat{q}(n) = \overline{\widehat{q}(-n)}, \quad n \in \mathbb{Z},$$
$$\omega(n) = \overline{\omega(-n)}, \quad n \in \mathbb{Z},$$

and as a consequence

$$\min_{\pm} \left| \gamma_n(q) \pm 2\sqrt{(\widehat{q}+\omega)(-n)(\widehat{q}+\omega)(n)} \right| = \left| \gamma_n(q) - 2\left|(\widehat{q}+\omega)(n)\right| \right|.$$

The proof of Theorem 1 is complete.

To prove Theorem 2 we firstly prove its Corollary. The formula (9) follows from [KpMh, Corollary 0.2 (2.6)], the Isospectral Theorem 5 and [MiMl6, Theorem C]. Also it can be proved directly as well as [KpMh, Corollary 0.2 (2.6)] using the estimates (5).

Further, to prove Theorem 2 it is sufficient to apply the asymptotic estimates (5), the properties (7) and (8) of the refined scales, and the formula (9):

- $$q \in H^{s,\varphi}_{1\text{-}per}(\mathbb{R},\mathbb{R}) \overset{(7)}{\Longrightarrow} q \in H^{s-\delta}_{1\text{-}per}(\mathbb{R},\mathbb{R}),\ \delta > 0 \overset{(5)}{\Longrightarrow} \gamma_n = 2\left|\widehat{q}(n)\right| + h^{1+2(s-\delta)-\varepsilon}(n)$$
$$\overset{(8)}{\Longrightarrow} \gamma_n = 2\left|\widehat{q}(n)\right| + h^{s,\varphi}(n) \Longrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s,\varphi};$$
- $$\{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s,\varphi} \overset{(8)}{\Longrightarrow} \{\gamma_n\}_{n=1}^{\infty} \in h^{s-\delta},\ \delta > 0 \overset{(9)}{\Longrightarrow} q \in H^{s-\delta}_{1\text{-}per}(\mathbb{R},\mathbb{R})$$
$$\overset{(5)}{\Longrightarrow} \gamma_n = 2\left|\widehat{q}(n)\right| + h^{1+2(s-\delta)-\varepsilon}(n) \overset{(8)}{\Longrightarrow} \gamma_n = 2\left|\widehat{q}(n)\right| + h^{s,\varphi}(n)$$
$$\Longrightarrow \{\widehat{q}(n)\}_{n\in\mathbb{Z}} \in h^{s,\varphi}(n).$$

Remark that due to arbitrary choice of $\delta > 0$ and $\varepsilon > 0$ we may choose them such that

$$1 + s - 2\delta - \varepsilon > 0.$$

The proof of Theorem 2 is complete. Now, we are ready to prove Theorem 3.

At first, note that from [KpMt, Theorem 1.2] we get the following asymptotic formulae for the lengths of spectral gaps:

$$\gamma_n(q) = 2\left|\widehat{q}(n)\right| + h^{1+s}(n) \quad \text{as} \quad q(x) \in H^{s}_{1\text{-}per}(\mathbb{R},\mathbb{R}),\ s \in [0,\infty), \tag{10}$$

which for the integer numbers $s \in \mathbb{Z}_+$ were proved by Marchenko and Ostrovskii [MrOs].

Using (9), (10) and (4) it is easy to prove the relationship

$$q(x) \in H^{s}_{1\text{-}per}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s}, \quad s \in (-1,\infty). \tag{11}$$

Sufficiency of Theorem 3. Let $q(x) \in H^{s,\varphi}_{1\text{-}per}(\mathbb{R},\mathbb{R})$. If $s \in (-1,0]$ then due to Theorem 2 we obtain that $\{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s,\varphi}$. If $s > 0$ then

$$q(x) \in H^{s,\varphi}_{1\text{-}per}(\mathbb{R},\mathbb{R}) \overset{(7)}{\hookrightarrow} H^{s-\delta}_{1\text{-}per}(\mathbb{R},\mathbb{R}),\ \delta > 0 \overset{(10)}{\Longrightarrow} \gamma_n(q) = 2\left|\widehat{q}(n)\right| + h^{1+s-\delta}(n)$$
$$\overset{(8)}{\Longrightarrow} \gamma_n(q) = 2\left|\widehat{q}(n)\right| + h^{s,\varphi}(n) \Longrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s,\varphi}.$$

Sufficiency is proved.

Necessity of Theorem 3. Let suppose that $\{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s,\varphi}$. If $s \in (-1,0]$ then from Theorem 2 we have that $q(x) \in H^{s,\varphi}_{1\text{-}per}(\mathbb{R},\mathbb{R})$. If $s > 0$ then

$$\{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s,\varphi} \overset{(8)}{\hookrightarrow} h^{s-\delta},\ \delta > 0 \overset{(11)}{\Longrightarrow} q(x) \in H^{s-\delta}_{1\text{-}per}(\mathbb{R},\mathbb{R}) \overset{(10)}{\Longrightarrow} \gamma_n(q) = 2\left|\widehat{q}(n)\right| + h^{1+s-\delta}(n)$$
$$\overset{(8)}{\Longrightarrow} \gamma_n(q) = 2\left|\widehat{q}(n)\right| + h^{s,\varphi}(n) \Longrightarrow q(x) \in H^{s,\varphi}_{1\text{-}per}(\mathbb{R},\mathbb{R}).$$

Necessity is proved.

The proof of Theorem 3 is complete.

4. Concluding remarks

In fact we can prove the following result: if $q(x) \in H^{-1+}_{1\text{-}per}(\mathbb{R},\mathbb{R})$ and

$$(1+2|k|)^s \ll w(k) \ll (1+2|k|)^{1+2s}, \quad s \in (-1,0],$$
$$(1+2|k|)^s \ll w(k) \ll (1+2|k|)^{1+s}, \quad s \in [0,\infty),$$

then

$$q(x) \in H^w_{1\text{-}per}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^w.$$

This result is not covered by the theorems of the preprint [DjMt3], because it does not require the weight function to be monotonic and submultiplicative.

Appendix

The complex Sobolev spaces $H^s_{1\text{-}per}(\mathbb{R})$, $s \in \mathbb{R}$ of 1-periodic functions and distributions over the real axis $\mathbb{R}$ are defined by means of their Fourier coefficients,

$$H^s_{1\text{-}per}(\mathbb{R}) := \left\{ f = \sum_{k\in\mathbb{Z}} \widehat{f}(k) e^{ik2\pi x} \;\Big|\; \| f \|_{H^s_{1\text{-}per}(\mathbb{R})} < \infty \right\},$$
$$\| f \|_{H^s_{1\text{-}per}(\mathbb{R})} := \left(\sum_{k\in\mathbb{Z}} \langle 2k\rangle^{2s} |\widehat{f}(k)|^2 \right)^{1/2}, \quad \langle k\rangle := 1+|k|,$$
$$\widehat{f}(k) := \langle f, e^{ik2\pi x}\rangle_{L^2_{1\text{-}per}(\mathbb{R})}, \quad k \in \mathbb{Z}.$$

By $\langle\cdot,\cdot\rangle_{L^2_{1\text{-}per}(\mathbb{R})}$ we denote a sesqui-linear form pairing the dual spaces $H^s_{1\text{-}per}(\mathbb{R})$ and $H^{-s}_{1\text{-}per}(\mathbb{R})$ with respect to $L^2_{1\text{-}per}(\mathbb{R})$, which (the sequi-linear form $\langle\cdot,\cdot\rangle_{L^2_{1\text{-}per}(\mathbb{R})}$) is an extension by continuity of the $L^2_{1\text{-}per}(\mathbb{R})$-inner product [Brz, GrGr]:

$$\langle f, g\rangle_{L^2_{1\text{-}per}(\mathbb{R})} := \int_0^1 f(x)\overline{g(x)}\,dx = \sum_{k\in\mathbb{Z}} \widehat{f}(k)\overline{\widehat{g}(k)} \quad \forall f, g \in L^2_{1\text{-}per}(\mathbb{R}).$$

It is useful to notice that

$$H^0_{1\text{-}per}(\mathbb{R}) \equiv L^2_{1\text{-}per}(\mathbb{R}).$$

By $H^{s+}_{1\text{-}per}(\mathbb{R})$, $s \in \mathbb{R}$ we denote an inductive limit of the Sobolev spaces $H^t_{1\text{-}per}(\mathbb{R})$ with $t > s$,

$$H^{s+}_{1\text{-}per}(\mathbb{R}) := \bigcup_{\varepsilon>0} H^{s+\varepsilon}_{1\text{-}per}(\mathbb{R}).$$

It is a topological space with an inductive topology.

In a similar fashion the Sobolev spaces $H^s_{\pm}[0,1]$, $s \in \mathbb{R}$ of 1-periodic/1-semiperiodic functions and distributions over the interval $[0,1]$ are defined:

$$H^s_{\pm}[0,1] := \left\{ f = \sum_{k\in\Gamma_{\pm}} \widehat{f}\left(\frac{k}{2}\right) e^{ik\pi x} \;\Big|\; \| f \|_{H^s_{\pm}[0,1]} < \infty \right\},$$
$$\| f \|_{H^s_{\pm}[0,1]} := \left(\sum_{k\in\Gamma_{\pm}} \langle k\rangle^{2s} \left|\widehat{f}\left(\frac{k}{2}\right)\right|^2 \right)^{1/2}, \quad \langle k\rangle = 1+|k|,$$
$$\widehat{f}\left(\frac{k}{2}\right) := \langle f(x), e^{ik\pi x}\rangle_{\pm}, \quad k \in \Gamma_{\pm}.$$

Here

$$\Gamma_+ \equiv 2\mathbb{Z} := \{k \in \mathbb{Z} \mid k \equiv 0 \pmod 2\},$$
$$\Gamma_- \equiv 2\mathbb{Z}+1 := \{k \in \mathbb{Z} \mid k \equiv 1 \pmod 2\},$$

and $\langle\cdot,\cdot\rangle_\pm$ are sesqui-linear forms pairing the dual spaces $H^s_\pm[0,1]$ and $H^{-s}_\pm[0,1]$ with respect to $L^2(0,1)$, which (the sesqui-linear forms $\langle\cdot,\cdot\rangle_\pm$) are extensions by continuity of the $L^2(0,1)$-inner product [Brz, GrGr]:

$$\langle f,g\rangle_\pm := \int_0^1 f(x)\overline{g(x)}\,dx = \sum_{k\in\Gamma_\pm} \widehat{f}\left(\frac{k}{2}\right)\overline{\widehat{g}\left(\frac{k}{2}\right)} \quad \forall f,g \in L^2(0,1).$$

It is obvious that

$$H^0_+[0,1] \equiv H^0_-[0,1] \equiv L^2(0,1).$$

We say that 1-periodic function or distribution $f(x)$ is *real-valued* if $\operatorname{Im} f(x) = 0$. Let us remind that

$$\operatorname{Re} f(x) := \frac{1}{2}(f(x) + \overline{f(x)}), \quad \operatorname{Im} f(x) := \frac{1}{2i}(f(x) - \overline{f(x)}),$$

(see, for an example, [Vld]). In terms of the Fourier coefficients we have

$$\operatorname{Im} f(x) = 0 \Leftrightarrow \widehat{f}(k) = \overline{\widehat{f}(-k)}, \quad k \in \mathbb{Z}.$$

Set

$$H^s_{1\text{-}per}(\mathbb{R},\mathbb{R}) := \left\{f(x) \in H^s_{1\text{-}per}(\mathbb{R}) \,|\, \operatorname{Im} f(x) = 0\right\},$$
$$H^{s+}_{1\text{-}per}(\mathbb{R},\mathbb{R}) := \left\{f(x) \in H^{s+}_{1\text{-}per}(\mathbb{R}) \,|\, \operatorname{Im} f(x) = 0\right\},$$
$$H^s_\pm([0,1],\mathbb{R}) := \left\{f(x) \in H^s_\pm[0,1] \,|\, \operatorname{Im} f(x) = 0\right\}.$$

Also we will need the Hilbert sequence spaces

$$h^s \equiv h^s(\mathbb{Z},\mathbb{C}), \quad s \in \mathbb{R}$$

of (two-sided) weighted sequences,

$$h^s := \left\{ a = \{a(k)\}_{k\in\mathbb{Z}} \;\middle|\; \|a\|_{h^s} := \left(\sum_{k\in\mathbb{Z}} \langle k\rangle^{2s}|a(k)|^2\right)^{1/2} < \infty \right\}, \quad \langle k\rangle = 1+|k|.$$

Note that

$$h^0 \equiv l^2(\mathbb{Z},\mathbb{C}),$$

and

$$a = \{a(k)\}_{k\in\mathbb{Z}} \in h^s,\ s \in \mathbb{R} \qquad \Rightarrow \qquad a(k) = o(|k|^{-s}), \quad k \to \pm\infty.$$

References

[Brz] Yu. M. Berezanskii, *Expantions in Eigenfunctions of Self-Adjoint Operators*, Transl. Math. Monographs, vol. 17, Amer. Math. Soc., Providence, R. I, 1968. (Russian edition: Naukova Dumka, Kiev, 1965)

[DjMt1] P. Djakov, B. Mityagin, *Spectral triangles of Schrödinger operators with complex potentials*, Selecta Math. (N.S.) **9** (2003), no. 4, 495–528.

[DjMt2] P. Djakov, B. Mityagin, *Instability zones of one-dimentional periodic Schrödinger and Dirac operatots*, Uspekhi Mat. Nauk **61** (2006), no. 4, 77–182. (Russian)

[DjMt3] P. Djakov, B. Mityagin, *Fourier method for one dimentional Schrödinger operators with singular periodic potentials*, arXiv:math.SP/0710.0237v1 1 Oct 2007, 1–39.

[DnSch2] N. Dunford, J. T. Schwartz, *Linear Operators*, Part II: *Spectral Theory. Self-Adjoint Operators in Hilbert Space*, Interscience, New York, London, 1963. (Russian edition: Mir, Moscow, 1966)

[Edv1] R. Edwards, *Fourier Series: A Modern Introduction*. Vol. I, Holt, Rinehard and Winston, Inc., New York-Montreal, Que.-London, 1967. (Russian edition: Mir, Moscow, 1985)

[Edv2] R. Edwards, *Fourier Series: A Modern Introduction*. Vol. II, Holt, Rinehard and Winston, Inc., New York-Montreal, Que.-London, 1967. (Russian edition: Mir, Moscow, 1985)

[GrGr] V. I. Gorbachuk, M. L. Gorbachuk, *Boundary Value Problems for Operator Differential Equations*, Kluwer, Dordrecht, etc., 1991. (Russian edition: Naukova Dumka, Kiev, 1984)

[HrMk] R. Hryniv, Ya. Mykytyuk, *1-D Schrödinger operators with periodic singular potentials*, Methods Funct. Anal. Topology **7** (2001), no. 4, 31–42.

[Hchs] H. Hochstadt, *Estimates of the stability intervals for Hill's equation*, Proc. Amer. Math. Soc. **14** (1963), 930–932.

[KpMt] T. Kappeler, B. Mityagin, *Estimates for periodic and Dirichlet eigenvalues of the Schrödinger operator*, SIAM J. Math. Anal. **33** (2001), no. 1, 113–152.

[KpMh] T. Kappeler, C. Möhr, *Estimates for periodic and Dirichlet eigenvalues of the Schrödinger operator with singular potentials*, J. Funct. Anal. **186** (2001), 62–91.

[Krt] E. Korotyaev, *Characterization of the spectrum of Schrödinger operators with periodic distributions*, Int. Math. Res. Not. **37** (2003), 2019–2031.

[MrOs] V. Marchenko, I. Ostrovskii, *A characterization of the spectrum of Hill's operator*, Matem. Sbornik **97** (1975), no. 4, 540–606. (Russian)

[McKTr] H. McKean, E. Trubowitz, *Hill's operators and hyperelliptic function theory in the presence of infinitely many branch points*, Comm. Pure Appl. Math. **29** (1976), no. 2, 143–226.

[MiMl1] V. Mikhailets, V. Molyboga, *Singular eigenvalue problems on the circle*, Methods Funct. Anal. Topology **10** (2004), no. 3, 44–53.

[MiMl2] V. Mikhailets, V. Molyboga, *Uniform estimates for the semi-periodic eigenvalues of the singular differential operators*, Methods Funct. Anal. Topology **10** (2004), no. 4, 30–57.

[MiMl3] V. Mikhailets, V. Molyboga, *The spectral problems over the periodic classes of distributions*, Preprint, Institute of Mathematics of NAS of Ukraine, Kyiv, 2004, 46 p. (Ukrainian)

[MiMl4] V. Mikhailets, V. Molyboga, *The perturbation of periodic and semiperiodic operators by Schwartz distributions*, Reports of NAS of Ukraine **7** (2006), 26 – 31. (Russian)

[MiMl5] V. Mikhailets, V. Molyboga, *Singularly perturbed periodic and semiperiodic differential operators*, Ukrainian Math. J. **59** (2007), no. 6, 785–797.

[MiMl6] V. Mikhailets, V. Molyboga, *One-dimensional Schrödinger operators with singular periodic potentials*, Methods Funct. Anal. Topology **14** (2008), no. 2, 184–200.

[MiMr] V. Mikhailets, A. Murach, *Interpolation with a function parameter and refined scale of spaces*, Methods Funct. Anal. Topology **14** (2008), no. 1, 81–100.

[MiSb] V. Mikhailets, A. Sobolev, *Common eigenvalue problem and periodic Schrödinger operators*, J. Funct. Anal. **165** (1999), no. 1, 150–172.

[Mlb1] V. Molyboga, *Estimates for periodic eigenvalues of the differential operator* $(-1)^m d^{2m}/dx^{2m} + V$ *with* V *– distribution*, Methods Funct. Anal. Topology **9** (2003), no. 2, 163–178.

[Mlb2] V. Molyboga, *Spectra of periodic problems with generalized functions*, Candidate Dissertation, Kyiv, 2005, 167 p. (Ukrainian)

[Mhr] C. Möhr, *Schrödinger Operators with Singular Potentials on the Circle: Spectral Analysis and Applications*, Thesis at the University of Zürich, 2001, 134 p.

[Psch] J. Pöschel, *Hill's potentials in weighted Sobolev spaces and their spectral gaps*, Preprint, Stuttgarter Mathematische Berichte, Stuttgart, 2004, 28 p.

[ReSi4] M. Reed, B. Simon, *Methods of Modern Mathematical Physics*: Vols 1-4, Academic Press, New York, etc., 1972–1978, V. 4: *Analysis of Operators*, 1972. (Russian edition: Mir, Moscow, 1982)

[SvSh] A. Savchuk, A. Shkalikov, *Sturm-Liouville operators with distribution potentials*, Tr. Mosk. Mat. Obs. **64** (2003), 159–212. (Russian) Engl. transl. in Trans. Moscow Math. Soc. (2003), 143–192.

[Snt] E. Seneta, *Regularly Varying Functions*, Lecture Notes in Math., vol. 508, Springer-Verlag, Berlin, 1976. (Russian edition: Nauka, Moscow, 1985)

[Vld] V. S. Vladimirov, *Generalized Functions in Mathematical Physics*, Nauka, Moscow, 1976, 280 p. (Russian)

Institute of Mathematics, National Academy of Science of Ukraine, Tereshchenkivska str., 3, Kyiv-4, Ukraine, 01601

E-mail address: mikhailets@imath.kiev.ua

Institute of Mathematics, National Academy of Science of Ukraine, Tereshchenkivska str., 3, Kyiv-4, Ukraine, 01601

E-mail address: molyboga@imath.kiev.ua