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# Nondifferentiable Functions of One Dimensional Semimartingales

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**Abstract**

In this paper we consider decompositions of processes of the form $Y = f(t, X_t)$ where $X$ is a one dimensional semimartingale, but $f$ is not required to be differentiable so Ito's formula does not apply.

First, in the case where $f(t, x)$ is independent of $t$, we show that requiring it to be locally Lipschitz continuous in $x$ is enough for an Ito style decomposition to apply. This decomposes $Y$ into a stochastic integral term and a term whose quadratic variation is well defined and has zero continuous part.

For the time dependent case we show that the same decomposition still holds under the additional conditions that the left and right derivatives of $f(t, x)$ in $x$ exist, it is right-continuous in $t$, and that locally its variation with respect to $t$ is integrable in $x$. In particular, in the continuous case this shows that $Y$ is a Dirichlet process. We furthermore prove that such processes satisfy a decomposition into continuous martingale and purely discontinuous terms, and a Doob-Meyer style decomposition.

## 1 Introduction

If $X$ is a real valued semimartingale and $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is any twice continuously differentiable function then Ito's formula gives the decomposition

$$f(t, X_t) = \int_0^t f_{,2}(s, X_{s-})\, dX_s + V_t \tag{1}$$

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where $V$ is a process of locally finite variation given by

$$V_t = f(0,X_0) + \frac{1}{2}\int_0^t f_{,22}(s,X_s)\,d[X]^c_s + \int_0^t f_{,1}(s,X_s)\,ds + \sum_{s\in(0,t]}\left(\Delta f(s,X_s) - f_{,2}(s,X_{s-})\Delta X_s\right). \tag{2}$$

In particular, this implies that $f(t,X_t)$ is a semimartingale. We have used the notation $\Delta X_t \equiv X_t - X_{t-}$ for the jumps of a cadlag process $X$, $[X]$ for the quadratic variation of a semimartingale $X$, and

$$[X]^c_t \equiv [X]_t - \sum_{s\le t}(\Delta X_s)^2$$

for the continuous part of $[X]$.

The main aim of this paper is to extend equation (1) to a class of functions $f$ that are not differentiable. We shall consider functions that are locally Lipschitz continuous in the variable $x$. Although we do not require differentiability, this Lipschitz condition is enough to ensure that $f$ will be differentiable in an almost-everywhere sense, so that equation (1) is well defined. For the case where $f$ is not time dependent, then this is enough for our decomposition (Theorem 2.5) to apply. However, as the second derivative of $f$ need not exist anywhere, equation (2) will not be well defined. In fact, the process $V$ need not even be locally of finite variation. Instead, it is shown that it satisfies some properties of finite variation processes, namely that its quadratic variation exists and is a pure jump process.

For time dependent functions, we do not require differentiability or even continuity in $t$. Instead, we just require $f(t,x)$ to be right-continuous and such that locally its variation in $t$ is integrable with respect to $x$. We also require the additional condition that $f$ has left and right derivatives in $x$. Then, the main decomposition result is stated in Theorem 2.7.

Alternative decompositions are given that apply to $f(t,X_t)$ and, more generally, to any process that can be written as a sum of a semimartingale and a process whose quadratic variation is a pure jump process. In Theorem 3.1 we show that there is a unique decomposition into continuous local martingale and purely discontinuous parts, and then in Theorem 3.2 we prove a Doob-Meyer style decomposition. These two decomposition results are extensions of well known results for semimartingales.

One motivation for looking at decompositions of $f(t,X_t)$ in the nondifferentiable case is to study one dimensional time inhomogeneous diffusions.

For example, suppose we have a martingale $X$ that satisfies an SDE of the form

$$dX_t = \sigma(t, X_t)\, dW_t \tag{3}$$

where $\sigma$ is Lipschitz continuous and $W$ is a Brownian motion. For any given initial value for $X_0$, it is well known that it has a unique solution ([15] V.11, [14] V Theorem 6, [9] 9.53). Furthermore, the process $X$ will satisfy the Markov property. Then, given any time $T > 0$ and any convex function $g : \mathbb{R} \to \mathbb{R}$ such that $g(X_T)$ is integrable, we can define

$$f(t, X_t) = \mathbb{E}\left[g(X_T) \mid \mathcal{F}_t\right]$$

for any time $t \leq T$. Such a situation occurs in finance when pricing options under a local volatility model. In this case, $X$ is the price of a traded asset and $g(X_T) = (X_T - k)_+$ is the payoff of a call option with expiry $T$ and strike price $k$. Then, $f(t, X_t)$ is the price of the option at time $t$ as a function of the underlying asset price. As is well known, the price of a call option is convex in the price $X_t$ of the asset and decreasing in $t$. This was proven by [1] and [12], although they required $\sigma$ to be continuously differentiable. In [11], this was proven under much weaker conditions — only the uniqueness of solutions to SDE (3) was required, and in [13] the result was extended to all strong Markov martingales satisfying an 'almost-continuity' condition. However, in general $f$ does not have to be differentiable in $x$ or continuous in $t$.

In order to study the dynamics of such processes further, it would be useful to derive conditions for $f(t, X_t)$ to be a martingale. In the case where $X$ is given by the SDE (3) and $f$ is twice continuously differentiable then this can be done by applying Ito's formula, and leads to the Stroock-Varadhan martingale problem studied in depth in [17]. For general martingale diffusions and nondifferentiable functions $f$ then this will not work. However, as mentioned above, $f(t, x)$ will be both convex in $x$ and decreasing in $t$. As convex functions are necessarily locally Lipschitz continuous with left and right derivatives everywhere, the results in this paper can be applied. For example, it is easy to show that all continuous martingale diffusions satisfy the previsible representation property (see Corollary 2.9). We shall continue these ideas further in a later paper.

Many authors have looked at extending equations (1) and (2) to different cases where $f(t, x)$ is not twice differentiable, so we shall briefly discuss some of the cases that are already known.

In the case where $f$ does not depend on $t$ and is a convex function of $x$ then the well known Ito-Meyer-Tanaka formula can be applied. For a general non-continuous semimartingale $X$ it gives

$$V_t = f(X_0) + \frac{1}{2}\int_{-\infty}^{\infty} L_t^x\, f''(dx) + \sum_{s\le t}\left(\Delta f(X_s) - f'(X_{s-})\Delta X_s\right).$$

See [14] Chapter IV, Theorem 70 or [9] Theorem 9.46. Here, $L_t^x$ is the semimartingale local time of $X$ at the level $x$. As $L_t^x$ is an increasing function of $t$, $V$ will also be increasing and in particular will have locally finite variation.

More generally, if $f(t,x)$ is convex in $x$ and right-continuous and increasing in $t$, then $V$ will be an increasing process. For the twice continuously differentiable case this follows from equation (2), and extends easily to the nondifferentiable case by approximating by smooth functions. However, this does not help in the case where $f$ is decreasing in $t$.

The case where $f(t,x)$ is once continuously differentiable is also well known. If $Y, Z$ are processes taking values in $\mathbb{R}^n$ with well defined quadratic variations and mutual covariations (in particular, if they are semimartingales) and $g, h : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable, then the covariation of $g(Y)$ and $h(Z)$ exists, and is given by

$$[g(Y),h(Z)]_t = \sum_{i,j=1}^{n}\int_0^t g_{,i}(Y_s)h_{,j}(Z_s)\,d[Y^i,Z^j]^c_s + \sum_{s\le t}\Delta g(Y_s)\Delta h(Z_s). \tag{4}$$

See remark (2) in [6]. If $Y$ is continuous then a simple application of the bilinearity of covariations gives

$$g(Y_t) = \sum_{i=1}^{n}\int_0^t g_{,i}(Y_{s-})\,dY^i_s + A_t \tag{5}$$

where $A$ is a process with zero quadratic variation. In our case, this shows that if $X$ is a continuous semimartingale and $f$ is continuously differentiable, then the process $V$ in equation (1) will have zero quadratic variation. In particular, $f(t,X_t)$ will decompose as a sum of a continuous semimartingale and a process of zero quadratic variation. Such processes were introduced by Follmer in [7] and are known as *Dirichlet* processes. The definition was extended to non-continuous processes in [2], where a Dirichlet process is defined as the sum of a semimartingale and a process with zero quadratic variation.

If $Y$ is not continuous, then the process $A$ in equation (5) will not be continuous, so its quadratic variation must be non-zero. However, the continuous part of its quadratic variation will still be zero, so $[A]$ will be a pure jump process,

$$[A]_t = \sum_{s \le t} (\Delta A_s)^2.$$

Therefore, if $f$ is continuously differentiable and $X$ is a non-continuous semimartingale then the process $V$ in equation (1) will have well defined quadratic variation whose continuous part is zero. Furthermore, it is shown in [2] that the class of Dirichlet processes is closed under $C^1$ transformations, so $f(t, X_t)$ will be a Dirichlet process.

Many authors have considered an extension of Dirichlet processes known as *weak Dirichlet* processes (see, for example, [8], [3], [16]). These are processes that can be expressed as a sum of a semimartingale and a process that has zero quadratic covariation with any continuous martingale. Certainly, in the case where $f(t, x)$ is convex in $x$ and locally of bounded variation in $t$ then the results of this paper will imply that $f(t, X_t)$ is a weak Dirichlet process, but this is a much weaker result.

In [4] they give a generalization of Ito's formula in the case where (1) the left derivative $\partial f/\partial t$ exists and is left-continuous in $t$ and (2) there is a decomposition $f = f_h + f_v$ such that $\partial f_h/\partial x$ exists, is continuous, and has left continuous and locally bounded left derivative, and the left derivative $\partial f_v/\partial x$ exists and has locally bounded variation in $(t, x)$.

Furthermore, there are also many generalizations of Ito's formula that can be obtained by placing restrictions on the semimartingale $X$. For example, if $f(t, x)$ is convex in $x$ and decreasing in $t$ and $X$ is a reversible semimartingale then we can reduce it to the case mentioned above where $f(t, x)$ is increasing in $t$.

We have briefly covered a few of the many extensions of Ito's formula that are already available, but none of these include the cases considered in this paper. In particular, they do not apply where $X$ is a one dimensional semimartingale and $f(t, x)$ is convex in $x$ and right-continuous and decreasing in $t$.

## 2 Functions of Semimartingales

In order to state our decomposition results we first need to define what we mean by the quadratic variation of a process. There are several different definitions of quadratic variations and covariations available in the literature,

although they all coincide for semimartingales. As we shall be concerned with processes that may not be semimartingales, we state precisely what we mean by the quadratic variation of general real valued processes.

We define the approximations to the quadratic variation by using partitions of $\mathbb{R}_+$. A (stochastic) partition $P$ of $\mathbb{R}_+$ is a sequence of stopping times $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots$ such that $\tau_n \to \infty$ as $n \to \infty$. Then for any partition $P$ we shall write the sequence it defines as $\tau_n^P$, so

$$0 = \tau_0^P \leq \tau_1^P \leq \tau_2^P \leq \cdots \uparrow \infty.$$

For any such partition $P$, we use $|P|$ to denote its maximum step size

$$|P| = \sup_{k\in\mathbb{N}} \|\tau_k^P - \tau_{k-1}^P\|_\infty.$$

Here, $\|U\|_\infty$ represents the supremum norm of a random variable $U$, that is, it is the smallest positive real number that is almost surely an upper bound for $|U|$. Given a stochastic process $X$ and a partition $P$ of $\mathbb{R}_+$, we shall use the notation

$$\delta_k^P X \equiv X_{\tau_k^P} - X_{\tau_{k-1}^P}.$$

We now define the quadratic variation and covariation as follows,

**Definition 2.1.** *Let $X$, $Y$ be real valued cadlag processes. Then their covariation $[X,Y]$, if it exists, is a cadlag process such that*

$$\lim_{|P|\to 0} \mathbb{P}\left(\sup_{n\in\mathbb{N},\tau_n^P\leq t}\left|\sum_{k=1}^n \delta_k^P X \delta_k^P Y - [X,Y]_{\tau_n^P}\right| > \epsilon\right) = 0$$

*for every $t, \epsilon > 0$, where the limit is taken over partitions $P$ of $\mathbb{R}_+$.*

*Given any cadlag process $X$ then its quadratic variation $[X]$, if it exists, is defined to be $[X] \equiv [X,X]$.*

This definition of quadratic covariations coincides with the standard ones for semimartingales, by [14], II Theorem 23.

It follows from the definition that the jumps of the process $[X,Y]$ are given by

$$\Delta[X,Y] = \Delta X \Delta Y.$$

Also, the quadratic variation $[X]$ must be an increasing process. Furthermore if the covariation $[X,Y]$ and quadratic variations $[X]$, $[Y]$ exist then so does $[X+Y, X+Y]$ (by bilinearity), and

$$[X,Y] = ([X+Y] - [X] - [Y])/2,$$

so $[X,Y]$ is a difference of increasing processes and therefore has locally finite variation. If $[X,Y]$ exists and is of locally finite variation then we define its continuous part by

$$[X,Y]^c_t = [X,Y]_t - \sum_{s\le t} \Delta X_s \Delta Y_s.$$

Similarly the continuous part of the quadratic variation $[X]^c \equiv [X,X]^c$ satisfies

$$[X]^c_t = [X]_t - \sum_{s\le t} (\Delta X_s)^2.$$

We shall say that a process $X$ has *zero continuous quadratic variation* if its quadratic variation $[X]$ exists and $[X]^c = 0$. When $X$ is a semimartingale, this coincides with the definition of *quadratic pure jump* semimartingales used by [14] (Section II page 71). It is well known that any process of locally finite variation has zero continuous quadratic variation ([14] II Theorem 26). Furthermore, a continuous semimartingale is locally of finite variation if and only if it has zero quadratic variation. To see that this is the case, use the decomposition $X = M + A$ of a continuous semimartingale into its continuous local martingale part $M$ and finite variation part $A$. Then, $X$ is of locally finite variation if and only if $M$ is constant or, equivalently, $[X] = [M] = 0$.

We therefore consider processes of zero continuous quadratic variation as a generalization of the locally finite variation processes. As semimartingales are sums of local martingales and locally finite variation processes, the processes that can be expressed as a sum of a local martingale (or semimartingale) and an adapted zero continuous quadratic variation process form a natural generalization of semimartingales.

**Definition 2.2.** *We say that a real valued process $X$ is a (non-continuous)* Dirichlet process *if it has a decomposition*

$$X = Y + V$$

*where $Y$ is a semimartingale and $V$ is a cadlag adapted process with zero continuous quadratic variation.*

In the case where $X$ is continuous then this reduces to the same definition of Dirichlet processes as used by Follmer in [7], and by many subsequent authors (see the comment following Theorem 3.1). In [2] the term 'non-continuous Dirichlet process' has already been used to refer to a process

that is the sum of a semimartingale and a (continuous) process of zero quadratic variation. However, this is too restrictive for our purposes, so we generalize slightly in the above definition to allow the zero continuous quadratic variation component to be non-continuous and for its quadratic variation $[V]$ to be non-zero, only requiring that $[V]^c = 0$. The definition of Dirichlet processes used in this paper will also differ slightly from that used by other authors due to the different definitions used for the quadratic variation.

Before moving on to the statements of the decomposition theorems, we first briefly mention some of the properties of locally finite variation processes that generalize to the zero continuous quadratic variation case. It is well known that the quadratic covariation $[X,Y]$ of a locally finite variation process $X$ and a semimartingale $Y$ has zero continuous part ([14] II Theorem 28), and it remains true if $Y$ is only assumed to have a well defined quadratic variation ([5] Proposition 2.5). This result extends to the processes of zero continuous quadratic variation, as the following lemma states.

**Lemma 2.3.** *Let $X$ be a process with zero continuous quadratic variation, and $Y$ be a process such that $[Y]$ exists. Then, the covariation $[X,Y]$ exists and satisfies*

$$[X,Y]_t = \sum_{s\leq t} \Delta X_s \Delta Y_s.$$

We shall leave the proof of this until Section 4. If we know that the quadratic covariation $[X,Y]$ exists, then Lemma 2.3 would follow from the Kunita-Watanabe inequality,

$$\int |H||K|\,|d[X,Y]| \leq \left(\int H\,d[X] \int K\,d[Y]\right)^{\frac{1}{2}}.$$

See [14] II Theorem 25 for the semimartingale case, and their proof will also apply to general cadlag processes whose quadratic variations and covariation exists. The main difficulty in proving Lemma 2.3 is in showing that the covariation does exist. A simple consequence of this lemma is that the class of processes with zero continuous quadratic variation is closed under taking linear combinations.

**Corollary 2.4.** *Let $X$ and $Y$ be processes with zero continuous quadratic variation. Then, $X+Y$ also has zero continuous quadratic variation.*

*Proof.* By Lemma 2.3, the covariation $[X,Y]$ exists and satisfies $[X,Y]^c = 0$. Therefore, from the bilinearity of Definition 2.1 of $[X,Y]$ in $X$ and $Y$, we

see that $[X+Y]=[X+Y,X+Y]$ exists and satisfies

$$[X+Y]^c=[X]^c+2[X,Y]^c+[Y]^c=0. \qquad \square$$

In order to generalize equation (1) to nondifferentiable functions, we need to use a generalization of the derivative of a function $f(x)$. We use the notation

$$\nabla_- f(x)=\liminf_{h\to 0}(f(x+h)-f(x))/h,$$
$$\nabla_+ f(x)=\limsup_{h\to 0}(f(x+h)-f(x))/h.$$

Then $-\infty\le\nabla_- f\le\nabla_+ f\le\infty$, and $f$ is differentiable at $x$ if and only if $\nabla_- f(x)=\nabla_+ f(x)$ is finite. Furthermore, if $f$ is Lipschitz continuous with constant $K$ on some neighbourhood of $x$ then $-K\le\nabla_- f(x)\le\nabla_+ f(x)\le K$. We now state our first decomposition result, the proof of which is left until Section 7.

**Theorem 2.5.** *Let $X$ be a semimartingale and $f:\mathbb{R}\to\mathbb{R}$ be a locally Lipschitz continuous function. Then,*

$$f(X_t)=\int_0^t\nabla_+ f(X_{s-})\,dX_s+V_t$$

*where $V$ is a process with zero continuous quadratic variation.*

In particular, this shows that $f(X)$ is a Dirichlet process. It is already known that the class of Dirichlet processes is closed under applying continuously differentiable functions (see [2]), so Theorem 2.5 gives a significant generalization of this in the case where $X$ is a one dimensional semimartingale.

We shall also state a decomposition for time-dependent functions. The variation of a function $f(t,x)$ as $t$ varies over the interval $[0,T]$ and $x$ is fixed shall be written as $\int_0^T|d_t f(t,x)|$. Also, the statement that $f(t,x)$ is locally Lipschitz continuous in $x$ means that for every bounded $S\subset\mathbb{R}_+\times\mathbb{R}$ there is a constant $K$ such that

$$|f(t,x)-f(t,y)|\le K\,|x-y|$$

whenever $(t,x)$ and $(t,y)$ are in $S$. We also use $\nabla_- f$ and $\nabla_+ f$ to represent the 'derivatives' with respect to $x$,

$$\nabla_- f(t,x)\equiv\liminf_{h\to 0}(f(t,x+h)-f(t,x))/h,$$
$$\nabla_+ f(t,x)\equiv\limsup_{h\to 0}(f(t,x+h)-f(t,x))/h.$$

Then, the class of functions that we shall be concerned with is as follows,

**Definition 2.6.** *We shall denote by $\mathcal{D}$ the set of functions $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that*

- *$f(t,x)$ is locally Lipschitz continuous in $x$ and cadlag in $t$,*
- *for every $K_0 < K_1 \in \mathbb{R}$ and $T \in \mathbb{R}_+$ then*

$$\int_{K_0}^{K_1}\int_0^T |d_t f(t,x)|\,dx < \infty,$$

- *the left and right derivatives of $f(t,x)$ with respect to $x$ exist everywhere.*

We now state the decomposition for $f(t,X_t)$ where $X$ is a semimartingale. The proof of this is left until Section 7.

**Theorem 2.7.** *Let $X$ be a semimartingale and $f \in \mathcal{D}$. Then,*

$$f(t,X_t) = \int_0^t \nabla_+ f(s,X_{s-})\,dX_s + V_t$$

*where $V$ has zero continuous quadratic variation.*

In particular, we see that $f(t,X_t)$ will be a Dirichlet process. Note that in the case where $f$ has no time dependence, then this theorem still requires $f(t,x)$ to have left and right derivatives everywhere in $x$. However, we know from Theorem 2.5 that this is not necessary. So we could ask whether this decomposition still holds if we drop the requirement for $f(t,x)$ to have left and right derivatives in $x$ in Definition 2.6 for $f \in \mathcal{D}$. It is not clear whether or not this is the case, but it would follow if it could be shown that $f$ would be differentiable in the necessary almost-everywhere case (by extending Theorem 6.3).

An immediate consequence of this result is that we can calculate the quadratic covariations of such functions of semimartingales.

**Corollary 2.8.** *Let $X,Y$ be semimartingales and $f,g \in \mathcal{D}$. Then the covariation of the processes $f(t,X_t)$ and $g(t,Y_t)$ exists and,*

$$\begin{aligned}[f(\cdot,X_\cdot),g(\cdot,Y_\cdot)]_t = &\int_0^t \nabla_+ f(s,X_{s-})\nabla_+ g(s,Y_{s-})\,d[X,Y]^c_s \\ &+ \sum_{s\le t} \Delta_s f(s,X_s)\Delta_s g(s,Y_s).\end{aligned} \tag{6}$$

*Proof.* First, by Theorem 2.7 we can write

$$f(t,X_t) = \int_0^t \nabla_+ f(s,X_{s-})\,dX_s + U_t,$$
$$g(t,Y_t) = \int_0^t \nabla_+ g(s,Y_{s-})\,dY_s + V_t$$

with $[U]^c = [V]^c = 0$. Lemma 2.3 gives us the following covariations.

$$\left[U, \int \nabla_+ g(s,Y_{s-})\,dY_s\right]^c = \left[\int \nabla_+ f(s,X_{s-})\,dX_s, V\right]^c = [U,V]^c = 0.$$

Then using the bilinearity of quadratic covariations together with the equality

$$\left[\int \nabla_+ f(s,X_{s-})\,dX_s, \int \nabla_+ g(s,Y_{s-})\,dY_s\right]^c$$
$$= \int \nabla_+ f(s,X_{s-})\nabla_+ g(s,Y_{s-})\,d[X,Y]^c_s$$

gives

$$[f(\cdot,X_\cdot), g(\cdot,Y_\cdot)]^c_t = \int_0^t \nabla_+ f(s,X_{s-})\nabla_+ g(s,Y_{s-})\,d[X,Y]^c_s.$$

Equation (6) follows directly from this. □

Another simple consequence of Theorem 2.7 is the following result for local martingales.

**Corollary 2.9.** *Let $X$ be a real valued continuous local martingale. If $f \in \mathcal{D}$ and $f(t,X_t)$ is a local martingale then,*

$$f(t,X_t) = f(0,X_0) + \int_0^t \nabla_+ f(s,X_s)\,dX_s.$$

*Proof.* As $X$ is continuous, it is a previsible process. So, $f(t,X_t)$ is a previsible local martingale and therefore is continuous. Writing

$$V_t = f(t,X_t) - f(0,X_0) - \int_0^t \nabla_+ f(s,X_s)\,dX_s$$

we see that $V$ is a continuous local martingale with $V_0 = 0$. Theorem 2.7 then tells us that it has zero quadratic variation, so $\mathbb{E}\left[V_t^2\right] \le \mathbb{E}\left[[V]_t\right] = 0$, giving the result. □

In particular, it is clear that if $f(t,x)$ is convex in $x$ and right-continuous and decreasing in $t$ then $f \in \mathcal{D}$, and it follows that one dimensional martingale diffusions satisy the previsible representation property.

## 3 Decompositions of Dirichlet Processes

In this section we shall state and prove two different decompositions of Dirichlet processes that are generalizations of well known semimartingale decompositions.

First, every semimartingale decomposes uniquely into a continuous local martingale part and a quadratic pure jump semimartingale part (In [9] Chapter VIII Definition 8.2 they use this decomposition in order to define the quadratic covariation). We can easily extend this to general non-continuous Dirichlet process (Definition 2.2).

**Theorem 3.1.** *Let $X$ be a Dirichlet process. Then, there is a unique decomposition*

$$X = M + A \tag{7}$$

*where $M$ is a continuous local martingale with $M_0 = 0$ and $A$ has zero continuous quadratic variation.*

*Proof.* Write $X = Y + Z$ where $Y$ is a semimartingale and $Z$ has zero continuous quadratic variation. As $Y$ is a semimartingale, it decomposes as $Y = N + V$ where $N$ is a local martingale and $V$ locally has finite variation. Therefore, $[V]^c = 0$. We now use Theorem 7.25 of [9] to decompose $N$ as $N = M + N'$ where $M$ is a continuous local martingale with $M_0 = 0$ and $N'$ is a purely discontinuous local martingale that satisfies $[N']^c = 0$. Decomposition (7) now follows by setting $A = N' + V + Z$.

It only remains to prove uniqueness. So suppose that we have two such decompositions $X = M + A = M' + A'$. Then, $M - M'$ is a continuous local martingale with zero quadratic variation, and so

$$\mathbb{E}\left[(M_t - M'_t)^2\right] \le \mathbb{E}\left[[M - M']_t\right] = 0. \qquad \square$$

In particular, if $X$ is a Dirichlet process according to Definition 2.2 and is continuous, then Theorem 3.1 shows that it can be expressed as a sum of a local martingale and a continuous process of zero continuous quadratic variation. So, $X$ decomposes as a sum of a continuous local martingale and a zero quadratic variation part. This shows that for continuous processes, Definition 2.2 coincides with the definition of Dirichlet processes used by Follmer in [7] and by other authors (up to slight differences in the definition of the quadratic variation).

Another well known result for semimartingales is the canonical decomposition into local martingale and previsible locally finite variation parts. This

requires $\sup_{s\le t}|X_s|$ to be locally integrable, and such processes are known as special semimartingales (see [14] III Theorem 33 or [9] Theorem 8.6). This can be considered as a generalization of the Doob-Meyer decomposition to more general semimartingales. We can extend this result to non-continuous Dirichlet processes. For any process $X$ we shall write $X_t^* \equiv \sup_{s\le t}|X_s|$ for its maximum process.

**Theorem 3.2.** *Let $X$ be a Dirichlet process such that $X^*$ is locally integrable. Then there exists a unique decomposition*

$$X = M + A \tag{8}$$

*where $M$ is a local martingale with $M_0 = 0$ and $A$ is a previsible process with zero continuous quadratic variation.*

*Proof.* First, uniqueness is easy. Suppose that there exists two such decompositions $X = M + A = M' + A'$. Then $M - M' = A' - A$ would be a previsible local martingale and so would be continuous. It would also have zero continuous quadratic variation and therefore, by the continuity of $M - M'$, its quadratic variation must be zero. So,

$$\mathbb{E}\left[(M_t - M_t')^2\right] \le \mathbb{E}\left[[M - M']_t\right] = 0$$

showing that $M = M'$ and $A = A'$.

In order to prove existence of the decomposition, we first use Theorem 3.1 to write $X = M^c + V$ where $M^c$ is a continuous local martingale with $M_0^c = 0$ and $V$ has zero continuous quadratic variation. By continuity $(M^c)^*$ is locally bounded and, in particular, must be locally integrable. So, $V^* \le (M^c)^* + X^*$ is also locally integrable. We shall show that $V$ satisfies the decomposition (8).

Write ${}^p\Delta V$ for the previsible projection of the process $\Delta V$ and set $H = \Delta V - {}^p\Delta V$. The idea is to apply Theorem 7.42 of [9] to show that there exists a local martingale $N$ satisfying $\Delta N = H$. In order to do this, we must first show that $H$ is a *thin* process. That is, we must show that it is zero everywhere except at a countable set of stopping times ([9], Definition 7.39).

There exists a sequence $(T_n)_{n\in\mathbb{N}}$ of strictly positive stopping times such that for every $n$, $T_n$ is either previsible or totally inaccessible, $[\![T_n]\!]\cap[\![T_m]\!] = \emptyset$ whenever $m \ne n$, and

$$\bigcup_{n\in\mathbb{N}} [\![T_n]\!]$$

contains all the times at which $V$ is discontinuous (see [9] Theorem 4.21). Then, we can write ${}^p\Delta V$ as

$$ {}^p\Delta V = \sum_{n=1}^{\infty} {}^p\left(1_{[\![T_n]\!]}\Delta V_{T_n}\right).$$

If $T_n$ is totally inaccessible then $\mathbb{P}\left(T = T_n\right) = 0$ for every previsible stopping time $T$, so the previsible projection of $1_{[\![T_n]\!]}\Delta V_{T_n}$ is zero. Therefore,

$$ {}^p\Delta V = \sum_{n=1}^{\infty} 1_{\{T_n \text{ is previsible}\}} 1_{[\![T_n]\!]} \mathbb{E}\left[\Delta V_{T_n}|\mathcal{F}_{T_n-}\right].$$

So $H$ is a thin process satisfying

$$ H = \sum_{n=1}^{\infty} 1_{[\![T_n]\!]}\left(\Delta V_{T_n} - 1_{\{T_n \text{ is previsible}\}}\mathbb{E}\left[\Delta V_{T_n}|\mathcal{F}_{T_n-}\right]\right).$$

Also $H$ has zero previsible projection, as ${}^pH = {}^p\Delta V - {}^{pp}\Delta V = 0$. Using the local integrability of $V^*$, Theorem 7.10 of [9] says that the process $t \mapsto \sqrt{\sum_{s\le t}\Delta V_s^2}$ is locally integrable. Therefore, Theorem 7.42 of [9] says that there exists a local martingale $N$ satisfying $\Delta N = H$.

We now use Theorem 7.25 of [9] to decompose $N$ as $N = N^c + N^d$ where $N^c$ is a continuous local martingale and $N^d$ is a purely discontinuous local martingale with $N^d = 0$. As $N^c$ is continuous, we have $\Delta N^d = \Delta N = \Delta V - {}^p\Delta V$.

Now define the process $A$ by

$$ V = N^d + A.$$

Then $A$ can be written as

$$ A_t = A_{t-} + \Delta A_t = A_{t-} + \Delta V_t - \Delta N_t^d = A_{t-} + ({}^p\Delta V)_t$$

which expresses it as the sum of a left-continuous and adapted (and therefore previsible) process and the previsible process ${}^p\Delta V$. So, $A$ is previsible. Furthermore, as $N^d$ is purely discontinuous its quadratic variation satisfies

$$ [N^d]_t = \sum_{s\le t}(\Delta N_s^d)^2.$$

In [9] this follows directly from Definition 7.29 of the quadratic variation, and their definition does indeed coincide with ours by their remark following Theorem 9.33. So, $N^d$ has zero continuous quadratic variation and, as $A = V - N^d$, we see that $A$ must also satisfy this property.

Decomposition (8) now follows by setting $M = M^c + N^d$. □

## 4 Zero Continuous Quadratic Variation

In this section we shall prove some simple results concerning processes with zero continuous quadratic variation. In particular , we shall prove Lemma 2.3. In the case where $X$ is continuous with zero quadratic variation then it would follow directly from the Cauchy-Schwarz inequality,

$$\left|\sum_{k=1}^{n}\delta_k^P X\delta_k^P Y\right| \le \left(\sum_{k=1}^{n}(\delta_k^P X)^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}(\delta_k^P Y)^2\right)^{\frac{1}{2}}.$$

If $X$ has zero quadratic variation and $Y$ has bounded quadratic variation, then $[X,Y]=0$ follows from this. If $X$ merely has zero continuous quadratic variation then the idea that we shall use is to take out the times at which $X$ jumps and then make use of the Cauchy-Schwarz inequality to obtain $[X,Y]^c = 0$. We start with the following, which says that the quadratic variation of $X$ is zero in the limit as we take out the times at which it jumps,

**Lemma 4.1.** *Let $X$ be a cadlag process with zero continuous quadratic variation. For every $\delta>0$ let $S_\delta$ be the jointly measurable set*

$$S_\delta = \{(t,\omega)\in\mathbb{R}_+\times\Omega : |\Delta X_t(\omega)|\ge\delta\}. \tag{9}$$

*Then, for every $T\in\mathbb{R}_+$ and $\epsilon>0$,*

$$\lim_{\delta\to0}\limsup_{|P|\to0}\mathbb{P}\left(\sum_{\tau_k^P\le T}1_{\left\{\rrbracket\tau_{k-1}^P,\tau_k^P\rrbracket\cap S_\delta=\emptyset\right\}}(\delta_k^P X)^2>\epsilon\right)=0, \tag{10}$$

*where the limit is taken for $P$ ranging over the partitions of $\mathbb{R}_+$.*

*Proof.* For every $\delta>0$ define the continuous functions $\theta_\delta,\phi_\delta:\mathbb{R}\to\mathbb{R}$ by

$$\begin{aligned}\theta_\delta(x)&=\min(\max(|x|/\delta-1,0),1)\\ \phi_\delta(x)&=1-\theta_\alpha(x)\end{aligned} \tag{11}$$

so that $\theta_\delta(x)=0$ for $|x|\le\delta$ and $\theta_\delta(x)=1$ for $|x|\ge2\delta$. We shall make use of the following limit,

$$\sup_{\tau_n^P\le T}\left|\sum_{k=1}^{n}\theta_\delta(\delta_k^P X)(\delta_k^P X)^2-\sum_{t\in(0,\tau_n^P]}\theta_\delta(\Delta X_t)(\Delta X_t)^2\right|\to0 \tag{12}$$

as $|P|\to 0$. This follows from the continuity of $\theta_\delta$ and the fact that for fine enough partitions, the summations reduce to a finite sum over the intervals containing times $t$ satisfying $|\Delta X_t|\geq\delta$. Using $\theta_\delta+\phi_\delta=1$ also gives

$$\begin{aligned}(13)\qquad &\sup_{\tau_n^P\leq T}\left|\sum_{k=1}^n\phi_\delta(\delta_k^PX)(\delta_k^PX)^2-\sum_{t\leq\tau_n^P}\phi_\delta(\Delta X_t)(\Delta X_t)^2\right|\\ \leq\ &\sup_{\tau_n^P\leq T}\left|\sum_{k=1}^n(\delta_k^PX)^2-\sum_{t\leq\tau_n^P}(\Delta X_t)^2\right|\\ +\ &\sup_{\tau_n^P\leq T}\left|\sum_{k=1}^n\theta_\delta(\delta_k^PX)(\delta_k^PX)^2-\sum_{t\leq\tau_n^P}\theta_\delta(\Delta X_t)(\Delta X_t)^2\right|.\end{aligned}$$

As $X$ has zero continuous quadratic variation, it satisfies $[X]_t=\sum_{s\leq t}(\Delta X_s)^2$. Therefore, Definition 2.1 of the quadratic variation shows that the first term on the right hand side of inequality (13) goes to 0 in probability as $|P|\to 0$. Also, limit (12) shows that the second term goes to zero. So,

$$(14)\qquad \sup_{\tau_n^P\leq T}\left|\sum_{k=1}^n\phi_\delta(\delta_k^PX)(\delta_k^PX)^2-\sum_{t\leq\tau_n^P}\phi_\delta(\Delta X_t)(\Delta X_t)^2\right|\to 0$$

in probability as $|P|\to 0$. As $\phi_\delta(x)=1$ for $|x|\leq\delta$ we have

$$1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S_\delta=\emptyset\}}\leq\phi_\delta(\delta_k^PX)$$

for fine enough partitions $P$. So, using limit (14),

$$\begin{aligned}&\limsup_{|P|\to 0}\mathbb{P}\left(\sum_{\tau_k^P\leq T}1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S_\delta=\emptyset\}}(\delta_k^PX)^2>\epsilon\right)\\ \leq\ &\limsup_{|P|\to 0}\mathbb{P}\left(\sum_{\tau_k^P\leq T}\phi_\delta(\delta_k^PX)(\delta_k^PX)^2>\epsilon\right)\\ \leq\ &\mathbb{P}\left(\sum_{t\leq T}\phi_\delta(\Delta X_t)(\Delta X_t)^2\geq\epsilon\right).\end{aligned}$$

Finally $\phi_\delta\to 0$ as $\delta\to 0$, so dominated convergence gives the result. □

In fact, limit (10) is a necessary and sufficient condition for $X$ to have zero continuous quadratic variation. However, we shall use more general sets than the $S_\delta$ defined by equation (9). We define $\mathcal{S}$ to be the collection of sets that can be expressed as the graphs of sequences of random times increasing to infinity.

$$\mathcal{S} = \left\{ \bigcup_{n=1}^{\infty} [\![T_n]\!] : \ T_n : \Omega \to \mathbb{R}_+ \cup \{\infty\} \text{ are measurable and } T_n \uparrow \infty \right\}. \tag{15}$$

By the debut theorem, a set $S \subseteq \mathbb{R}_+ \times \Omega$ is in $\mathcal{S}$ if and only if it is jointly measurable and only contains finitely many times in any finite time interval. That is, $\{t \le T : (t,\omega) \in S\}$ is finite for every $T \ge 0$ and $\omega \in \Omega$. In particular, the sets $S_\delta$ defined by equation (9) are in $\mathcal{S}$. We now give a sufficient condition for $[X,Y]^c = 0$ to be satisfied.

**Lemma 4.2.** *Let $X$ and $Y$ be cadlag processes. Suppose that*

$$\inf_{S\in\mathcal{S}} \limsup_{|P|\to 0} \mathbb{P}\left( \sum_{\tau_k^P \le T} 1_{\{]\!]\tau_{k-1}^P, \tau_k^P]\!] \cap S = \emptyset\}} |\delta_k^P X \delta_k^P Y| > \epsilon \right) = 0 \tag{16}$$

*for every $T \ge 0$ and $\epsilon > 0$. Here, the limit is taken as $P$ ranges over the partitions of $\mathbb{R}_+$.*

*Then, the quadratic covariation $[X,Y]$ exists and $[X,Y]^c = 0$.*

*Proof.* First, we note that for every $S \in \mathcal{S}$ and $T \ge 0$,

$$\begin{aligned} \sum_{t<T} |\Delta X_t \Delta Y_t| \le & \sum_{t<T} 1_{\{t\in S\}} |\Delta X_t \Delta Y_t| \\ & + \liminf_{|P|\to 0} \sum_{\tau_k^P \le T} 1_{\{]\!]\tau_{k-1}^P, \tau_k^P]\!] \cap S = \emptyset\}} |\delta_k^P X \delta_k^P Y|. \end{aligned}$$

The right hand side of this expression must be finite for some $S \in \mathcal{S}$, otherwise the probability in equation (16) would always be 1. Therefore, we can define the locally finite variation process

$$A_t = \sum_{s\le t} \Delta X_s \Delta Y_s.$$

To complete the proof of the lemma we only need to show that $[X,Y] = A$. Choose any $\delta > 0$ and let $\theta_\delta, \phi_\delta$ be the functions defined by equations (11). As $\theta_\delta(x)$ is continuous, and equal to 0 for $|x| \le \delta$ we get the limit

$$\sup_{\tau_n^P \le T} \left| \sum_{k=1}^{n} \theta_\delta(\delta_k^P X) \delta_k^P X \delta_k^P Y - \sum_{t \le \tau_n^P} \theta_\delta(\Delta X_t) \Delta X_t \Delta Y_t \right| \to 0 \tag{17}$$

as $|P| \to 0$. This limit follows from the fact that for fine enough partitions, the sum of terms for which $\theta_\delta(\delta_k^P X) \neq 0$ reduces to a finite sum of terms for which $[\![\tau_{k-1}^P, \tau_k^P]\!]$ contains a time $t$ satisfying $|\Delta X_t| \geq \delta$.

As $\theta_\delta + \phi_\delta = 1$ we also have the inequality

$$\begin{aligned}
&\sup_{\tau_n^P \leq T} \left| \sum_{k=1}^n \delta_k^P X \delta_k^P Y - A_{\tau_n^P} \right| \\
\leq &\sup_{\tau_n^P \leq T} \left| \sum_{k=1}^n \theta_\delta(\delta_k^P X) \delta_k^P X \delta_k^P Y - \sum_{t \leq \tau_n^P} \theta_\delta(\Delta X_t) \Delta X_t \Delta Y_t \right| \\
+ &\sum_{\tau_k^P \leq T} \phi_\delta(\delta_k^P X) |\delta_k^P X \delta_k^P Y| + \sum_{t \leq T} \phi_\delta(\Delta X_t) |\Delta A_t|
\end{aligned}$$

By limit (17), the first term on the right hand side goes to 0 as $|P| \to 0$. Now let $S_\delta \in \mathcal{S}$ be the set defined by equation (9). As $\phi_\delta(x) = 0$ for $|x| \geq 2\delta$, for fine enough partitions $P$ we have $\phi_\delta(\delta_k^P X) \leq 1_{\{[\![\tau_{k-1}^P, \tau_k^P]\!] \cap S_{3\delta} = \emptyset\}}$. Therefore,

$$\begin{aligned}
&\limsup_{|P| \to 0} \mathbb{P}\left( \sup_{\tau_n^P \leq T} \left| \sum_{k=1}^n \delta_k^P X \delta_k^P Y - A_{\tau_n^P} \right| > \epsilon \right) \\
\leq &\limsup_{|P| \to 0} \mathbb{P}\left( \sum_{\tau_k^P \leq T} 1_{\{[\![\tau_{k-1}^P, \tau_k^P]\!] \cap S_{3\delta} = \emptyset\}} |\delta_k^P X \delta_k^P Y| \geq \epsilon/2 \right) \\
+ &\mathbb{P}\left( \sum_{t \leq T} \phi_\delta(\Delta X_t) |\Delta A_t| \geq \epsilon/2 \right).
\end{aligned}$$

As $A$ has locally finite variation and $\phi_\delta(x)x \to 0$ as $\delta \to 0$, dominated convergence shows that the second term on the right hand side goes to 0 as

$\delta \to 0$. Therefore, choosing any $S \in \mathcal{S}$,

$$
\begin{aligned}
(18)\ &\limsup_{|P|\to 0} \mathbb{P}\left( \sup_{\tau_n^P \le T} \left| \sum_{k=1}^{n} \delta_k^P X \delta_k^P Y - A_{\tau_n^P} \right| > \epsilon \right) \\
&\le \liminf_{\delta \to 0} \limsup_{|P|\to 0} \mathbb{P}\left( \sum_{\tau_k^P \le T} 1_{\left\{ ]\!]\tau_{k-1}^P, \tau_k^P]\!] \cap S_{2\delta} = \emptyset \right\}} |\delta_k^P X \delta_k^P Y| \ge \epsilon/2 \right) \\
&\le \limsup_{|P|\to 0} \mathbb{P}\left( \sum_{\tau_k^P \le T} 1_{\left\{ ]\!]\tau_{k-1}^P, \tau_k^P]\!] \cap S = \emptyset \right\}} |\delta_k^P X \delta_k^P Y| \ge \epsilon/4 \right) \\
&+ \liminf_{\delta \to 0} \limsup_{|P|\to 0} \mathbb{P}\left( \sum_{\tau_k^P \le T} 1_{\left\{ ]\!]\tau_{k-1}^P, \tau_k^P]\!] \cap S_{2\delta} = \emptyset, ]\!]\tau_{k-1}^P, \tau_k^P]\!] \cap S \ne \emptyset \right\}} |\delta_k^P X \delta_k^P Y| \ge \epsilon/4 \right)
\end{aligned}
$$

As $S \cap ]\!]0, T]\!]$ is a finite set, the sum in the second term on the right hand side of this inequality only contains finitely many terms, so we can take the limit,

$$
\begin{aligned}
&\limsup_{|P|\to 0} \sum_{\tau_k^P \le T} 1_{\left\{ ]\!]\tau_{k-1}^P, \tau_k^P]\!] \cap S_{2\delta} = \emptyset, ]\!]\tau_{k-1}^P, \tau_k^P]\!] \cap S \ne \emptyset \right\}} |\delta_k^P X \delta_k^P Y| \\
&\le \sum_{t \le T} 1_{\{t \in S, |\Delta X_t| \le 2\delta\}} |\Delta A_t|.
\end{aligned}
$$

Dominated convergence says that this goes to 0 as $\delta \to 0$. So the last term in inequality (18) is 0, giving

$$
\begin{aligned}
&\limsup_{|P|\to 0} \mathbb{P}\left( \sup_{\tau_n^P \le T} \left| \sum_{k=1}^{n} \delta_k^P X \delta_k^P Y - A_{\tau_n^P} \right| > \epsilon \right) \\
&\le \limsup_{|P|\to 0} \mathbb{P}\left( \sum_{\tau_k^P \le T} 1_{\left\{ ]\!]\tau_{k-1}^P, \tau_k^P]\!] \cap S = \emptyset \right\}} |\delta_k^P X \delta_k^P Y| \ge \epsilon/4 \right)
\end{aligned}
$$

Finally, by the condition of the lemma, taking the infimum over all $S \in \mathcal{S}$ gives 0 on the right hand side, so $[X, Y] = A$. □

Putting together the previous two lemmas gives the following necessary and sufficient condition for a process to have zero continuous quadratic variation.

**Lemma 4.3.** *Let $X$ be a cadlag pocess. Then it has zero continuous quadratic variation if and only if*

$$\inf_{S\in\mathcal{S}}\limsup_{|P|\to 0}\mathbb{P}\left(\sum_{\tau_k^P\le T}1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\}}(\delta_k^P X)^2>\epsilon\right)=0 \tag{19}$$

*for every $T\ge 0$ and $\epsilon>0$.*

*Proof.* If $X$ has zero continuous quadratic variation then equation (19) follows from Lemma 4.1. Conversely, if equation (19) is satisfied then Lemma 4.2 with $Y=X$ implies that $X$ has zero continuous quadratic variation. □

Finally for this section, we use the Cauchy-Schwarz inequality to prove Lemma 2.3.

*Proof of Lemma 2.3.* Choose any $S\in\mathcal{S}$. Then, the Cauchy-Schwarz inequality gives

$$\begin{aligned}&\sum_{\tau_k^P\le T}1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\}}|\delta_k^P X\delta_k^P Y|\\&\le\left(\sum_{\tau_k^P\le T}1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\}}(\delta_k^P X)^2\right)^{\frac{1}{2}}\left(\sum_{\tau_k^P\le T}(\delta_k^P Y)^2\right)^{\frac{1}{2}}.\end{aligned}$$

As the quadratic variation $[Y]$ is well defined we can take limits as $|P|\to 0$,

$$\begin{aligned}&\limsup_{|P|\to 0}\mathbb{P}\left(\sum_{\tau_k^P\le T}1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\}}|\delta_k^P X\delta_k^P Y|>\epsilon\right)\\&\le\limsup_{|P|\to 0}\mathbb{P}\left([Y]_T\sum_{\tau_k^P\le T}1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\}}(\delta_k^P X)^2\ge\epsilon^2\right)\\&\le\limsup_{|P|\to 0}\mathbb{P}\left(\sum_{\tau_k^P\le T}1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\}}(\delta_k^P X)^2\ge\epsilon^2/K\right)+\mathbb{P}\left([Y]_T\ge K\right)\end{aligned}$$

for all $\epsilon,K>0$. As $X$ has zero continuous quadratic variation then Lemma 4.3 says that the first term on the right hand side of this inequality goes to 0 if we take the infimum over all $S\in\mathcal{S}$. Then, taking the limit as $K\to\infty$, we see that the second term on the right hand side also vanishes. So, the result follows from Lemma 4.2. □

## 5 Functions of Dirichlet Processes

In this section we shall state and prove the most general decomposition result of this paper for processes of the form $f(t,X_t)$, which we will use to prove Theorem 2.7. As we shall require $f(t,x)$ to be differentiable with respect to $x$ in an almost-everywhere sense, we start by defining

$$\begin{aligned}\text{(20)}\qquad \text{diff}(f) &= \{(t,x)\in\mathbb{R}_+\times\mathbb{R} : f(t,x) \text{ is differentiable in } x\}\\ &= \{(t,x)\in\mathbb{R}_+\times\mathbb{R} : \nabla_- f(t,x) = \nabla_+ f(t,x)\}\,.\end{aligned}$$

We also define the subset of $\mathbb{R}_+\times\mathbb{R}$ at which $f(t,x)$ is differentiable with respect to $x$ in a rather strong sense.

$$\begin{aligned}\text{(21)}\quad \text{diffC}(f) &= \left\{(t,x) : \lim_{s\to t,\ y,z\to x}(f(s,z)-f(s,y))/(z-y) \text{ exists}\right\}\\ &\subseteq \text{diff}(f).\end{aligned}$$

Here, the limit is taken over all $s\in\mathbb{R}_+$ and $y,z\in\mathbb{R}$ with $y\neq z$. Alternatively, $\text{diffC}(f)$ is the set of points at which $\nabla_+ f$ is continuous.

Recall that in Definition 2.6 of $f\in\mathcal{D}$ we required $f(t,x)$ to have left and right derivatives in $x$. As we shall not need this property in this section we make the following definition.

**Definition 5.1.** *We shall use $\mathcal{D}_0$ to represent the functions $f:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ such that $f(t,x)$ is locally Lipschitz continuous in $x$ and cadlag in $t$, and such that for every $K_0<K_1\in\mathbb{R}$ and $T\in\mathbb{R}_+$ then*

$$\text{(22)}\qquad \int_{K_0}^{K_1}\int_0^T |d_t f(t,x)|\,dx<\infty.$$

So $\mathcal{D}\subset\mathcal{D}_0$, and $f\in\mathcal{D}$ if and only if $f\in\mathcal{D}_0$ and $f(t,x)$ has left and right derivatives everywhere with respect to $x$. We now state the decomposition result.

**Theorem 5.2.** *Let $X=Y+Z$ where $Y$ is a semimartingale and $Z$ is a cadlag adapted process with zero continuous quadratic variation. Let $f\in\mathcal{D}_0$ satisfy*

$$\text{(23)}\qquad \int 1_{\{(t,X_t)\notin\text{diff}(f)\}}\,d[X]^c_t=0.$$

*and,*

$$\text{(24)}\qquad \iint 1_{\{(t,x)\notin\text{diffC}(f),\mathbb{P}(X_t=x)>0\}}\,|d_t f(t,x)|\,dx=0.$$

*Then,*

$$f(t, X_t) = \int_0^t \nabla_+ f(s, X_{s-})\, dY_s + V_t \tag{25}$$

*where $V$ is a process with zero continuous quadratic variation.*

Note that by applying the same result with $-f$ in place of $f$, we could equally well have used $\nabla_- f$ for the integrand in equation (25). Also, it follows from the properties of quadratic variations that for a semimartingale $Y$, the process $\int \xi\, dY$ has zero continuous quadratic variation if and only if

$$\int 1_{\{\xi \neq 0\}}\, d[Y]^c = 0.$$

Therefore, equation (23) is equivalent to the statement that the integral in equation (25) only differs by a process with zero continuous quadratic variation when $\nabla_+ f$ is replaced by $\nabla_- f$, explaining the need for this condition. We shall also show later, in Theorem 7.3, that equation (24) is not needed in the case where $X$ is a semimartingale.

The remainder of this section is dedicated to proving Theorem 5.2. Throughout this section, given any partition $P$ of $\mathbb{R}_+$, we shall suppose that we have a sequence of stopping times $(\sigma_k^P)_{k\in\mathbb{N}}$ satisfying

$$\begin{aligned} &\tau_{k-1}^P \leq \sigma_k^P \leq \tau_k^P, \\ &\sigma_k^P > \tau_{k-1}^P \text{ whenever } \tau_k^P > \tau_{k-1}^P \end{aligned} \tag{26}$$

for each $k$. We shall make the choice of which $\sigma_k^P$ to use later. Then we can define a new partition $\tilde{P}$ by

$$\tau_k^{\tilde{P}} = \begin{cases} \tau_{k/2}^P, & \text{if } k \text{ is even}, \\ \sigma_{(k+1)/2}^P, & \text{if } k \text{ is odd}. \end{cases} \tag{27}$$

We shall also pick a simple previsible process $\zeta$, according to the following definition.

**Definition 5.3.** *We define the* simple previsible *processes to be linear combinations of process of the form*

$$t \mapsto A 1_{\{t>T\}}$$

*where $T \in \mathbb{R}_+$ and $A$ is a bounded $\mathcal{F}_T$-measurable random variable.*

Then our approach to proving Theorem 5.2 is to split $\delta_k^P V$ into separate parts.

$$\begin{aligned}(28)\quad \delta_k^P V &= \left(f(\tau_k^P, X_{\tau_k^P}) - f(\sigma_k^P, X_{\tau_k^P}) + f(\sigma_k^P, X_{\tau_{k-1}^P}) - f(\tau_{k-1}^P, X_{\tau_{k-1}^P})\right)\\ &+ \left(f(\sigma_k^P, X_{\tau_k^P}) - f(\sigma_k^P, X_{\tau_{k-1}^P}) - \zeta_{\sigma_k^P}\delta_k^P X\right)\\ &+ \left(\zeta_{\sigma_k^P}\delta_k^P X - \int_{\tau_{k-1}^P}^{\tau_k^P} \nabla_+ f(t, X_{t-})\, dY_t\right).\end{aligned}$$

We shall look at each of the three terms on the right hand side in turn, and show that their contribution to the continuous part of the quadratic variation of $V$ can be made arbitrarily small (by making suitable choices of $\sigma_k^P$ and $\zeta$). Then we will finish off the proof by appealing to Lemma 4.3.

We start by showing that the contribution to the continuous part of the quadratic variation coming from the first term on the right hand side of (28) is zero. The idea is that we smooth out the time increments of $f$ by making use of the following identity

$$(29)\qquad g(y) = \frac{1}{a}\int_{y-a}^{y} (a - y + x)g'(x)\, dx + \frac{1}{a}\int_{y-a}^{y} g(x)\, dx$$

which is an application of integration by parts, and applies for every absolutely continuous function $g$ and every $a > 0$.

In the following lemma we do not require $X$ to satisfy any properties other than it being real valued and cadlag. In particular, we do not require it to have finite quadratic variation, so equation (23) is not used. As $f(t, x)$ is jointly continuous in $x$ and cadlag in $t$, there can only be countable many times $t$ at which it is not continuous. Therefore, it is possible to list them $t_1, t_2, t_3, \ldots$, and we can take $S_r = \{t_1, t_2, \ldots, t_r\}$ in the following lemma.

**Lemma 5.4.** *Let $X$ be a cadlag process and $f \in \mathcal{D}_0$ satisfy equation (24). Also, let $(S_r)_{r\in\mathbb{N}}$ be an increasing sequence of finite subsets of $\mathbb{R}_+$ that eventually includes every time $t$ at which $f(t, x)$ is not continuous. For every partition $P$ set*

$$A_k^P \equiv f(\tau_k^P, X_{\tau_k^P}) - f(\sigma_k^P, X_{\tau_k^P}) + f(\sigma_k^P, X_{\tau_{k-1}^P}) - f(\tau_{k-1}^P, X_{\tau_{k-1}^P}).$$

*Then for any $T \geq 0$,*

$$\lim_{r\to\infty} \limsup_{|P|\to 0} \sum_{\tau_k^P \leq T} 1_{\{]\tau_{k-1}^P, \tau_k^P]\cap S_r = \emptyset\}} \left(A_k^P\right)^2 = 0.$$

*Proof.* We first fix a path of $X$ by choosing any $\omega \in \Omega$ and treat $X_t \equiv X_t(\omega)$ as a deterministic process.

For any $u < v \in \mathbb{R}_+$ and $x \in \mathbb{R}$, we shall use the notation

$$\delta_{u,v} f(x) \equiv f(v,x) - f(u,x)$$

so, in particular,

$$A_k^P = \delta_{\sigma_k^P, \tau_k^P} f(X_{\tau_k^P}) + \delta_{\tau_{k-1}^P, \sigma_k^P} f(X_{\tau_{k-1}^P}). \tag{30}$$

Then, choosing any $a > 0$, we can use identity (29) with $g(x) = (\delta_{u,v} f(x))^2$ to get

$$\begin{aligned}
(\delta_{u,v} f(y))^2 &= \frac{2}{a} \int_{y-a}^{y} (a - y + x) \left(\delta_{u,v} \nabla_+ f(x)\right) \left(\delta_{u,v} f(x)\right) \, dx \\
&+ \frac{1}{a} \int_{y-a}^{y} \left(\delta_{u,v} f(x)\right)^2 \, dx \\
&= \frac{1}{a} \int_{y-a}^{y} \int_{u}^{v} \left(2(a - y + x) \delta_{u,v} \nabla_+ f(x) + \delta_{u,v} f(x)\right) \, d_t f(t,x) \, dx.
\end{aligned}$$

If $(\tau_{k-1}^P, \tau_k^P] \cap S_r = \emptyset$ then we can combine this with equation (30),

$$\begin{aligned}
(A_k^P)^2 &\leq 2 \left(\delta_{\sigma_k^P, \tau_k^P} f(X_{\tau_k^P})\right)^2 + 2 \left(\delta_{\tau_{k-1}^P, \sigma_k^P} f(X_{\tau_{k-1}^P})\right)^2 \\
&= \int_{-\infty}^{\infty} \int_{\tau_{k-1}^P}^{\tau_k^P} h_{r,a}^P(t,x) \, d_t f(t,x) \, dx
\end{aligned} \tag{31}$$

where we have defined the function $h_{r,a}^P(t,x)$ by

$$\begin{aligned}
h_{r,a}^P(t,x) &= \frac{2}{a} \sum_{k=1}^{\infty} 1_{\{t \in (\sigma_k^P, \tau_k^P], x \in (X_{\tau_k^P} - a, X_{\tau_k^P}), (\tau_{k-1}^P, \tau_k^P] \cap S_r = \emptyset\}} \\
&\quad \times \left(2(a - X_{\tau_k^P} + x) \delta_{\sigma_k^P, \tau_k^P} \nabla_+ f(x) + \delta_{\sigma_k^P, \tau_k^P} f(x)\right) \\
&+ \frac{2}{a} \sum_{k=1}^{\infty} 1_{\{t \in (\tau_{k-1}^P, \sigma_k^P], x \in (X_{\tau_{k-1}^P} - a, X_{\tau_{k-1}^P}), (\tau_{k-1}^P, \tau_k^P] \cap S_r = \emptyset\}} \\
&\quad \times \left(2(a - X_{\tau_{k-1}^P} + x) \delta_{\tau_{k-1}^P, \sigma_k^P} \nabla_+ f(x) + \delta_{\tau_{k-1}^P, \sigma_k^P} f(x)\right)
\end{aligned}$$

If we let $n$ be the maximum integer such that $\tau_n^P \leq T$ and sum inequality

(31) over $1 \le k \le n$ then we get

$$\sum_{\tau_k^P \le T} 1_{\{(\tau_{k-1}^P, \tau_k^P] \cap S_r = \emptyset\}} \left(A_k^P\right)^2 \le \int_{-\infty}^{\infty} \int_0^{\tau_n^P} h_{r,a}^P(t,x)\, d_t f(t,x)\, dx \tag{32}$$

$$\le \int_{-\infty}^{\infty} \int_0^T |h_{r,a}^P(t,x)|\, |d_t f(t,x)|\, dx.$$

For every $(t,x)$, the following inequality also follows from the definition of $h_{r,a}^P(t,x)$,

$$|h_{r,a}^P(t,x)| \le 1_{\{t \notin S_r\}} 1_{\{X_s - a \le x \le X_s\}} \left(4|\delta_{u,v} \nabla_+ f(x)| + 2|\delta_{u,v} f(x)|/a\right), \tag{33}$$

where $s, u, v$ are in the range $[t - |P|, t + |P|]$.

As $f(t,x)$ is locally Lipschitz in $x$, $\nabla_+ f$ will be bounded on any bounded set, and in particular will be bounded by some constant $K$ on the support of $1_{\{t \le T\}} h_{r,a}^P(t,x)$. As the sets $S_r$ eventually include every time at which $f(t-,x) \ne f(t,x)$, we can take limits in inequality (33),

$$\limsup_{r \to \infty} \limsup_{|P| \to 0} |h_{r,a}^P(t,x)| \le g_a(t,x),$$

$$g_a(t,x) \equiv 8K 1_{\{x \in [X_{t-} - a, X_{t-}] \cup [X_t - a, X_t]\}} 1_{\{(t,x) \notin \mathrm{diffC}(f)\}}.$$

So we can use bounded convergence in (32)

$$\limsup_{r \to \infty} \limsup_{|P| \to 0} \sum_{\tau_k^P \le T} 1_{\{(\tau_{k-1}^P, \tau_k^P] \cap S_r = \emptyset\}} \left(A_k^P\right)^2$$

$$\le \int_{-\infty}^{\infty} \int_0^T g_a(t,x)\, |d_t f(t,x)|\, dx.$$

Now take the limit as $a$ goes to 0 and again use bounded convergence,

$$\limsup_{r \to \infty} \limsup_{|P| \to 0} \sum_{\tau_k^P \le T} 1_{\{(\tau_{k-1}^P, \tau_k^P] \cap S_r = \emptyset\}} \left(A_k^P\right)^2$$

$$\le 8K \int_{-\infty}^{\infty} \int_0^T 1_{\{x = X_t, (t,x) \notin \mathrm{diffC}(f)\}}\, |d_t f(t,x)|\, dx.$$

To complete the proof we only need to show that the right hand side of this inequality is 0 for almost every $\omega \in \Omega$. However, if we take the expected value of the integral on the right hand side then it becomes

$$8K \int_{-\infty}^{\infty} \int_0^T \mathbb{P}\left(X_t = x\right) 1_{\{(t,x) \notin \mathrm{diffC}(f)\}}\, |d_t f(t,x)|\, dx$$

which is zero by equation (24). □

This lemma is the only part in the proof of Theorem 5.2 where equations (22) and (24) are required.

We now move on to bounding the contribution to the continuous part of the quadratic variation of $V$ coming from the second term on the right hand side of (28), for which we require the following lemma. It is here that we need to choose the stopping times $\sigma_k^P$, which we do by using the optional section theorem ([9] Theorem 4.7).

Recall that $\mathcal{S}$ was defined by equation (15). That is, it is the set of jointly measurable sets that are finite over finite time intervals for every $\omega\in\Omega$.

**Lemma 5.5.** *Let $X=Y+Z$ where $Y$ is a semimartingale and $Z$ is a cadlag adapted process with zero continuous quadratic variation. Also, let $\xi$ be any non-negative optional process uniformly bounded by some $K\in\mathbb{R}_+$. For every partition $P$ set*

$$B_k^P\equiv\xi_{\sigma_k^P}\left((X_{\tau_k^P}-X_{\sigma_k^P})^2+(X_{\sigma_k^P}-X_{\tau_{k-1}^P})^2\right).$$

*Then, for every $\delta>0$ we can choose the stopping times $\sigma_k^P$ satisfying inequalities (26) such that*

$$\begin{aligned}&\inf_{S\in\mathcal{S}}\limsup_{|P|\to0}\mathbb{P}\left(\sum_{\tau_k^P\le T}1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\}}B_k^P\ge\epsilon\right)\\&\le\mathbb{P}\left(\int_0^T\left(K1_{\{|\xi_s|>\delta\}}+\delta\right)d[X]_s^c\ge\epsilon\right)\end{aligned}$$

*for all $\epsilon,T>0$.*

*Proof.* First, by Theorem 3.1, we may suppose that $Y$ is a continuous local martingale. Let us write

$$\begin{aligned}C_k^P&\equiv\xi_{\sigma_k^P}\left((Y_{\tau_k^P}-Y_{\sigma_k^P})^2+(Y_{\sigma_k^P}-Y_{\tau_{k-1}^P})^2\right),\\D_k^P&\equiv\xi_{\sigma_k^P}\left((Z_{\tau_k^P}-Z_{\sigma_k^P})^2+(Z_{\sigma_k^P}-Z_{\tau_{k-1}^P})^2\right).\end{aligned}$$

Choosing any $S \in \mathcal{S}$ the triangle inequality gives,

$$\begin{aligned}
&\left(\sum_{k=1}^{n} 1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\}} B_k^P\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k=1}^{n} C_k^P\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{n} 1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\}} D_k^P\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k=1}^{n} C_k^P\right)^{\frac{1}{2}} + \left(K\sum_{k=1}^{2n} 1_{\left\{]\!]\tau_{k-1}^{\tilde{P}},\tau_k^{\tilde{P}}]\!]\cap S=\emptyset\right\}} (\delta_k^{\tilde{P}} Z)^2\right)^{\frac{1}{2}}
\end{aligned}$$

where $\tilde{P}$ is the partition defined by equation (27). If we choose any $\epsilon' < \epsilon$ and set $\epsilon'' = K^{-1}(\sqrt{\epsilon} - \sqrt{\epsilon'})^2$ this gives

$$\begin{aligned}
&\mathbb{P}\left(\sum_{\tau_k^P \leq T} 1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\}} B_k^P \geq \epsilon\right) \\
&\leq \mathbb{P}\left(\sum_{\tau_k^P \leq T} C_k^P \geq \epsilon'\right) + \mathbb{P}\left(\sum_{\tau_k^P \leq T} 1_{\left\{]\!]\tau_{k-1}^{\tilde{P}},\tau_k^{\tilde{P}}]\!]\cap S=\emptyset\right\}} (\delta_k^{\tilde{P}} Z)^2 \geq \epsilon''\right)
\end{aligned}$$

As $Z$ has zero continuous quadratic variation, Lemma 4.3 says that the second term on the right hand side vanishes if we let $|P|$ go to zero and take the infimum over all $S \in \mathcal{S}$,

(34)
$$\inf_{S\in\mathcal{S}} \limsup_{|P|\to 0} \mathbb{P}\left(\sum_{\tau_k^P \leq T} 1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\}} B_k^P \geq \epsilon\right) \leq \limsup_{|P|\to 0} \mathbb{P}\left(\sum_{\tau_k^P \leq T} C_k^P \geq \epsilon'\right).$$

This simplifies the problem to the case of a continuous local martingale.

We now make a choice for the stopping times $\sigma_k^P$. By the optional section theorem, this can be done such that inequalities (26) are satisfied and

$$\mathbb{P}\left(\xi_{\sigma_k^P} > \delta\right) \leq \mathbb{P}\left(\forall t \in (\tau_{k-1}^P, \tau_k^P],\ \xi_t > \delta\right) + 2^{-k}|P|.$$

We are furthermore free to choose $\sigma_k^P = \tau_k^P$ whenever $\xi_{\sigma_k^P} > \delta$. Also, by the Debut theorem we can define the stopping times

$$\tilde{\sigma}_\tau^P = \inf\left\{t \in (\tau_{k-1}^P, \tau_k^P] : \xi_t \leq \delta\right\} \cup \left\{\tau_k^P\right\}.$$

By our choice of $\sigma_k^P$ and $\tilde{\sigma}_k^P$, the following holds outside of a set of probability at most $2^{-k}|P|$

$$\begin{aligned}
C_k^P &\leq \delta\left((Y_{\tau_k^P} - Y_{\sigma_k^P})^2 + (Y_{\sigma_k^P} - Y_{\tau_{k-1}^P})^2\right) + K1_{\{\xi_{\sigma_k^P} > \delta\}}(Y_{\tau_k^P} - Y_{\tau_{k-1}^P})^2 \\
&\leq \delta\left((\delta_{2k}^{\tilde{P}}Y)^2 + (\delta_{2k-1}^{\tilde{P}}Y)^2\right) + K(Y_{\tilde{\sigma}_k^P} - Y_{\tau_{k-1}^P})^2 \\
&= \delta(\delta_{2k}^{\tilde{P}}Y)^2 + \delta(\delta_{2k-1}^{\tilde{P}}Y)^2 + 2K\int_{\tau_{k-1}^P}^{\tilde{\sigma}_k^P}(Y_s - Y_{\tau_{k-1}^P})\,dY_s + K\int_{\tau_{k-1}^P}^{\tilde{\sigma}_k^P} d[Y]_s
\end{aligned}$$

where $\tilde{P}$ is the partition defined by equation (27). Noting that $\xi_s > \delta$ whenever $s \in (\tau_{k-1}^P, \tilde{\sigma}_k^P)$ this inequality gives

$$C_k^P \leq \delta(\delta_{2k}^{\tilde{P}}Y)^2 + \delta(\delta_{2k-1}^{\tilde{P}}Y)^2 + K\int_{\tau_{k-1}^P}^{\tau_k^P} 1_{\{\xi_s > \delta\}}\,d[Y]_s + 2K\int_{\tau_{k-1}^P}^{\tau_k^P}\alpha_s^P\,dY_s \tag{35}$$

outside of a set with probability at most $2^{-k}|P|$ and with $\alpha^P$ being the previsible process

$$\alpha_s^P = \sum_{k=1}^{\infty} 1_{\{s \in (\tau_{k-1}^P, \tilde{\sigma}_k^P]\}}(Y_s - Y_{\tau_{k-1}^P}).$$

The continuity of $Y$ implies that $\alpha^P \to 0$ as $|P| \to 0$, so bounded convergence for stochastic integration gives

$$\sup_{t \leq T}\left|\int_0^t \alpha_s^P\,dY_s\right| \to 0$$

in probability as $|P| \to 0$. So summing inequality (35) over $k$ and taking the limit as $|P| \to 0$ gives

$$\begin{aligned}
&\limsup_{|P|\to 0}\mathbb{P}\left(\sum_{\tau_k^P \leq T} C_k^P \geq \epsilon'\right) \\
&\leq \limsup_{|P|\to 0}\mathbb{P}\left(\delta\sum_{\tau_k^{\tilde{P}} \leq T}(\delta_k^{\tilde{P}}Y)^2 + K\int_0^T 1_{\{\xi_s > \delta\}}\,d[Y]_s > \tilde{\epsilon}\right) + \limsup_{|P|\to 0}\sum_{k=0}^{\infty} 2^{-k}|P| \\
&\leq \mathbb{P}\left(\int_0^T \left(\delta + K1_{\{\xi_s > \delta\}}\right)\,d[Y]_s \geq \tilde{\epsilon}\right)
\end{aligned}$$

where $\tilde{\epsilon}$ is any real number in the range $0 < \tilde{\epsilon} < \epsilon'$. The result now follows from combining this with inequality (34) and letting $\tilde{\epsilon}$ increase to $\epsilon$. □

We use this result to bound the contribution to the continuous part of the quadratic variation of $V$ coming from the second term on the right hand side of (28).

**Lemma 5.6.** *Let $X = Y + Z$ where $Y$ is a semimartingale and $Z$ is an adapted cadlag process with zero continuous quadratic variation. Also let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(t,x)$ is cadlag in $t$ and Lipschitz continuous in $x$. Choosing any bounded optional process $\zeta$ and any $h > 0$ we set*

$$\xi_t \equiv \sup_{0<|a|\leq h} \left| \left( f(t, X_t + a) - f(t, X_t) \right) / a - \zeta_t \right|.$$

*Also, for every partition $P$ set*

$$C_k^P \equiv f(\sigma_k^P, X_{\tau_k^P}) - f(\sigma_k^P, X_{\tau_{k-1}^P}) - \zeta_{\sigma_k^P} \delta_k^P X.$$

*Then, for any $\delta > 0$ we can choose the stopping times $\sigma_k^P$ satisfying inequalities (26) such that*

$$\inf_{S\in\mathcal{S}} \limsup_{|P|\to 0} \mathbb{P}\left( \sum_{t_k^P \leq T} 1_{\{]\tau_{k-1}^P, \tau_k^P] \cap S = \emptyset\}} (C_k^P)^2 \geq \epsilon \right)$$
$$\leq \mathbb{P}\left( 2 \int_0^T (1_{\{\xi_t > \delta\}} K^2 + \delta^2)\, d[X]_t^c \geq \epsilon \right)$$

*for any $\epsilon, T > 0$, where $K \in \mathbb{R}$ is any upper bound for $\xi$.*

*Proof.* First note that we can restrict $a$ to the rational numbers in the definition of $\xi$, so it is the supremum of a countable set of optional processes and therefore is itself optional.

For every partition $P$ set

$$a_k^P \equiv X_{\tau_k^P} - X_{\sigma_k^P},$$
$$b_k^P \equiv X_{\tau_{k-1}^P} - X_{\sigma_k^P}.$$

Then we can rewrite $C_k^P$ as

$$C_k^P = 1_{\{a_k^P \neq 0\}} \left( \left( f(\sigma_k^P, X_{\sigma_k^P} + a_k^P) - f(\sigma_k^P, X_{\sigma_k^P}) \right) / a_k^P - \zeta_{\sigma_k^P} \right) a_k^P$$
$$- 1_{\{b_k^P \neq 0\}} \left( \left( f(\sigma_k^P, X_{\sigma_k^P} + b_k^P) - f(\sigma_k^P, X_{\sigma_k^P}) \right) / b_k^P - \zeta_{\sigma_k^P} \right) b_k^P$$

In particular, if $|a_k^P|$ and $|b_k^P|$ are both smaller than $h$ then

$$|C_k^P| \leq \xi_{\sigma_k^P} \left( \left| X_{\tau_k^P} - X_{\sigma_k^P} \right| + \left| X_{\sigma_k^P} - X_{\tau_{k-1}^P} \right| \right)$$

and so

$$(C_k^P)^2 \le B_k^P \equiv 2\xi^2_{\sigma_k^P}\left(\left(X_{\tau_k^P}-X_{\sigma_k^P}\right)^2+\left(X_{\sigma_k^P}-X_{\tau_{k-1}^P}\right)^2\right). \tag{36}$$

So, if we let $S\in\mathcal{S}$ include all the times $t$ for which $|\Delta X_t|\ge h$ then inequality (36) will hold whenever $[\![\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset$ and $\tau_k^P\le T$ for all fine enough partitions $P$. Therefore

$$\mathbb{P}\left(\sum_{t_k^P\le T}1_{\left\{[\![\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\right\}}(C_k^P)^2\ge\epsilon\right)\le\mathbb{P}\left(\sum_{t_k^P\le T}1_{\left\{[\![\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\right\}}B_k^P\ge\epsilon\right)$$

in the limit as $|P|\to 0$. The result now follows by applying Lemma 5.5 with $2\xi^2$ in place of $\xi$, $2K^2$ in place of $K$, and $2\delta^2$ in place of $\delta$. □

We now bound the contribution to the continuous part of the quadratic variation of $V$ coming from the third term on the right hand side of (28).

**Lemma 5.7.** *Let $X=Y+Z$ where $Y$ is a semimartingale and $Z$ is a cadlag adapted process with zero continuous quadratic variation. Given any uniformly bounded and previsible process $\eta$ and simple previsible process $\zeta$ set,*

$$D_k^P\equiv\zeta_{\sigma_k^P}\delta_k^P X-\int_{\tau_{k-1}^P}^{\tau_k^P}\eta_s\,dY_s$$

*for all partitions $P$ of $\mathbb{R}_+$. Then,*

$$\begin{aligned}&\inf_{S\in\mathcal{S}}\limsup_{|P|\to 0}\mathbb{P}\left(\sum_{\tau_k^P\le T}1_{\left\{[\![\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\right\}}\left(D_k^P\right)^2\ge\epsilon\right)\\&\le\mathbb{P}\left(\int_0^T(\zeta_s-\eta_s)^2\,d[X]^c\ge\epsilon\right)\end{aligned}$$

*for all $\epsilon,T>0$.*

*Proof.* First, as $\zeta$ is simple previsibe, it is piecewise constant and there are only finitely many times at which it is not continuous. So, we can restrict to those $S\in\mathcal{S}$ that contain all the discontinuity times of $\zeta$. In that case we have

$$1_{\left\{[\![\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\right\}}\zeta_{\sigma_k^P}=1_{\left\{[\![\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\right\}}\zeta_{\tau_{k-1}^P+}.$$

So,

$$\begin{aligned}1_{\{]\tau_{k-1}^P,\tau_k^P]\cap S=\emptyset\}}D_k^P &= 1_{\{]\tau_{k-1}^P,\tau_k^P]\cap S=\emptyset\}}\zeta_{\sigma_k^P}\delta_k^P Z \\ &\quad + 1_{\{]\tau_{k-1}^P,\tau_k^P]\cap S=\emptyset\}}\left(\zeta_{\tau_{k-1}^P+}\delta_k^P Y - \int_{\tau_{k-1}^P}^{\tau_k^P}\xi_s\,dY_s\right) \\ &= 1_{\{]\tau_{k-1}^P,\tau_k^P]\cap S=\emptyset\}}\zeta_{\sigma_k^P}\delta_k^P Z + 1_{\{]\tau_{k-1}^P,\tau_k^P]\cap S=\emptyset\}}\delta_k^P U\end{aligned}$$

where $U$ is the process $U=\int(\zeta-\xi)\,dY$. Then the triangle inequality gives

$$\begin{aligned}\left(\sum_{k=1}^n 1_{\{]\tau_{k-1}^P,\tau_k^P]\cap S=\emptyset\}}(D_k^P)^2\right)^{\frac{1}{2}} &\le \left(\sum_{k=1}^n 1_{\{]\tau_{k-1}^P,\tau_k^P]\cap S=\emptyset\}}K^2(\delta_k^P Z)^2\right)^{\frac{1}{2}} \\ &\quad + \left(\sum_{k=1}^n 1_{\{]\tau_{k-1}^P,\tau_k^P]\cap S=\emptyset\}}(\delta_k^P U)^2\right)^{\frac{1}{2}}\end{aligned}$$

where $K$ is any upper bound for $\zeta$. Now choose any $\epsilon,T>0$ and $\epsilon'\in(0,\epsilon)$, and set $\epsilon''=K^{-2}(\sqrt{\epsilon}-\sqrt{\epsilon'})^2$. Then the above inequality gives

$$\begin{aligned}&\mathbb{P}\left(\sum_{\tau_k^P\le T}1_{\{]\tau_{k-1}^P,\tau_k^P]\cap S=\emptyset\}}(D_k^P)^2\ge\epsilon\right) \\ &\le \mathbb{P}\left(\sum_{\tau_k^P\le T}1_{\{]\tau_{k-1}^P,\tau_k^P]\cap S=\emptyset\}}(\delta_k^P Z)^2\ge\epsilon''\right) \\ &\quad + \mathbb{P}\left(\sum_{\tau_k^P\le T}1_{\{]\tau_{k-1}^P,\tau_k^P]\cap S=\emptyset\}}(\delta_k^P U)^2\ge\epsilon'\right)\end{aligned}\tag{37}$$

As $Z$ has zero continuous quadratic variation, Lemma 4.3 says that the first term on the right hand side of this inequality vanishes if we take the limit as $|P|\to 0$ and then take the infimum over $S\in\mathcal{S}$. Also, the definition of

the quadratic variation $[U]$ gives

$$\begin{aligned}
&\limsup_{|P|\to 0}\mathbb{P}\left(\sum_{\tau_k^P\le T}1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\}}(\delta_k^P U)^2\ge\epsilon'\right)\\
&\le\mathbb{P}\left([U]_T-\sum_{t\le T}1_{\{t\in S\}}(\Delta U_t)^2\ge\epsilon'\right)\\
&=\mathbb{P}\left([U]_T^c+\sum_{t\le T}1_{\{t\notin S\}}(\Delta U_t)^2\ge\epsilon'\right)
\end{aligned}$$

Then, as we can let $S$ increase to include arbitrarily many of the discontinuity times of $U$,

$$\inf_{S\in\mathcal{S}}\limsup_{|P|\to 0}\mathbb{P}\left(\sum_{\tau_k^P\le T}1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\}}(\delta_k^P U)^2\ge\epsilon'\right)\le\mathbb{P}\left([U]_T^c\ge\epsilon'\right).$$

Substituting this into inequality (37),

$$\inf_{S\in\mathcal{S}}\limsup_{|P|\to 0}\mathbb{P}\left(\sum_{\tau_k^P\le T}1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\}}\left(D_k^P\right)^2\ge\epsilon\right)\le\mathbb{P}\left([U]_T^c\ge\epsilon'\right).$$

The result now follows by letting $\epsilon'$ increase to $\epsilon$ and noting that

$$[U]^c=\int(\zeta-\eta)^2\,d[Y]^c=\int(\zeta-\eta)^2\,d[X]^c.$$ □

Finally, for this section, we put together the results of Lemmas 5.4, 5.6 and 5.7 to prove Theorem 5.2.

*Proof of Theorem 5.2.* We need to show that $V$ has zero continuous quadratic variation. By localization, it is enough to do this in the case where $X$ is uniformly bounded. Also, we only need to show that $V$ has zero continuous quadratic variation over each finite time horizon. Therefore, we only need to consider the values of $f$ on an arbitrarily large but bounded subset of $\mathbb{R}_+\times\mathbb{R}$. As $f(t,x)$ is Lipschitz-continuous in $x$ on any bounded set, we may suppose that $f(t,x)$ is Lipschitz-continuous in $x$, with coefficient $L$.

Let $\eta$ be the previsible process

$$\eta_t=\nabla_+ f(t,X_{t-}),$$

so that $|\eta|$ is uniformly bounded by $L$. Also pick any simple previsible process $\zeta$ such that $|\zeta| \leq L$. For any $h > 0$ we also define the process

$$\xi_t^h \equiv \sup_{0<|a|\leq h} \left|\left(\left(f(t, X_t + a) - f(t, X_t)\right)/a - \zeta_t\right|,$$

which is bounded by $2L$. Then, according to equation (28) we have,

$$\delta_k^P V = A_k^P + C_k^P + D_k^P$$

with

$$\begin{aligned}
A_k^P &= f(\tau_k^P, X_{\tau_k^P}) - f(\sigma_k^P, X_{\tau_k^P}) + f(\sigma_k^P, X_{\tau_{k-1}^P}) - f(\tau_{k-1}^P, X_{\tau_{k-1}^P}),\\
C_k^P &= f(\sigma_k^P, X_{\tau_k^P}) - f(\sigma_k^P, X_{\tau_{k-1}^P}) - \zeta_{\sigma_k^P}\delta_k^P X,\\
D_k^P &= \zeta_{\sigma_k^P}\delta_k^P X - \int_{\tau_{k-1}^P}^{\tau_k^P} \eta_s \, dY_s.
\end{aligned}$$

In particular,

$$(\delta_k^P V)^2 \leq 3(A_k^P)^2 + 3(C_k^P)^2 + 3(D_k^P)^2.$$

So for any partition $P$ and set $S \in \mathcal{S}$,

$$\begin{aligned}
&\mathbb{P}\left(\sum_{\tau_k^P \leq T} 1_{\{]\tau_{k-1}^P, \tau_k^P] \cap S = \emptyset\}} (\delta_k^P V)^2 \geq \epsilon\right)\\
&\leq \mathbb{P}\left(\sum_{\tau_k^P \leq T} 1_{\{]\tau_{k-1}^P, \tau_k^P] \cap S = \emptyset\}} (A_k^P)^2 \geq \epsilon/3\right)\\
&+ \mathbb{P}\left(\sum_{\tau_k^P \leq T} 1_{\{]\tau_{k-1}^P, \tau_k^P] \cap S = \emptyset\}} (C_k^P)^2 \geq \epsilon/3\right)\\
&+ \mathbb{P}\left(\sum_{\tau_k^P \leq T} 1_{\{]\tau_{k-1}^P, \tau_k^P] \cap S = \emptyset\}} (D_k^P)^2 \geq \epsilon/3\right)
\end{aligned}$$

for all $\epsilon, T > 0$. By Lemma 5.4 the first term on the right hand side vanishes if we lake the limit as $|P| \to 0$ and then take the infimum over all $S \in \mathcal{S}$. So, by applying Lemmas 5.6 and 5.7 respectively to the second and third

terms on the right hand side, we can choose the stopping times $\sigma_k^P$ so that

$$\begin{aligned}(38)\qquad &\inf_{S\in\mathcal{S}}\limsup_{|P|\to 0}\mathbb{P}\left(\sum_{\tau_k^P\le T}1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\}}(\delta_k^P V)^2\ge\epsilon\right)\\&\le\mathbb{P}\left(2\int_0^T(1_{\{\xi_t^h>\delta\}}4L^2+\delta^2)\,d[X]_t^c\ge\epsilon/3\right)\\&+\mathbb{P}\left(\int_0^T(\zeta_t-\eta_t)^2\,d[X]_t^c\ge\epsilon/3\right)\end{aligned}$$

for any $\delta>0$. Also, whenever $(t,X_t)\in\mathrm{diff}(f)$ then the definition of $\xi^h$ gives

$$\xi_t^h\to|\nabla_+f(t,X_t)-\zeta_t|$$

as $h\to 0$. By equation (23), this limit holds almost everywhere with respect to the measure $\int_0^T\cdot\,d[X]^c$. Combining this with the inequality $1_{\{\xi^h>\delta\}}\le\delta^{-2}(\xi^h)^2$, we can take limits as $h\to 0$ in inequality (38),

$$\begin{aligned}(39)\qquad &\inf_{S\in\mathcal{S}}\limsup_{|P|\to 0}\mathbb{P}\left(\sum_{\tau_k^P\le T}1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\}}(\delta_k^P V)^2\ge\epsilon\right)\\&\le\mathbb{P}\left(2\int_0^T(4\delta^{-2}L^2(\nabla_+f(t,X_t)-\zeta_t)^2+\delta^2)\,d[X]_t^c\ge\epsilon/3\right)\\&+\mathbb{P}\left(\int_0^T(\zeta_t-\eta_t)^2\,d[X]_t^c\ge\epsilon/3\right).\end{aligned}$$

As the $\sigma$-algebra generated by the simple previsible processes is the whole of the previsible $\sigma$-algebra, a simple application of the monotone class lemma shows that for every bounded previsible process $\eta$ there exists a sequence of simple previsible processes $\zeta^n$ satisfying

$$\mathbb{P}\left(\int_0^T(\zeta_t^n-\eta_t)^2\,d[X]_t^c\ge\epsilon\right)\to 0$$

as $n\to\infty$ for every $\epsilon>0$. Furthermore, if $\eta$ is bounded by $L$ then $\zeta^n$ can also be chosen to be bounded by $L$. So, we can substitute $\zeta^n$ for $\zeta$ in the

right hand side of inequality (39) and take limits,

$$\begin{aligned}
&\inf_{S\in\mathcal{S}}\limsup_{|P|\to 0}\mathbb{P}\left(\sum_{\tau_k^P\le T}1_{\{]\!]\tau_{k-1}^P,\tau_k^P]\!]\cap S=\emptyset\}}(\delta_k^P V)^2\ge\epsilon\right)\\
&\le\mathbb{P}\left(2\int_0^T(4\delta^{-2}L^2(\nabla_+f(t,X_t)-\eta_t)^2+\delta^2)\,d[X]_t^c\ge\epsilon/3\right)\\
&=\mathbb{P}\left(2\delta^2[X]_T^c\ge\epsilon/3\right).
\end{aligned}$$

This last equality holds because $\eta_t=\nabla_+f(t,X_t)$ whenever $\Delta X_t=0$. The result now follows by letting $\delta$ decrease to 0 and applying Lemma 4.3. □

## 6 Almost Everywhere Differentiability

In this section we will show that for any $f\in\mathcal{D}$ then $f(t,x)$ is differentiable with respect to $x$ in an almost-everywhere sense, which shall be used to prove Theorem 2.7.

For every $a\in\mathbb{R}\setminus\{0\}$ we use $\nabla_a$ to represent the finite difference operator

$$\nabla_a f(t,x)\equiv\left(f(t,x+a)-f(t,x)\right)/a.$$

Also, for any $f\in\mathcal{D}_0$, we use $f^-(t,x)$ to represent its left limit in $t$,

$$f^-(t,x)\equiv\begin{cases}\lim_{s\uparrow\uparrow t}f(s,x), & \text{if } t>0,\\ f(0,x), & \text{if } t=0.\end{cases}$$

Then, we have the following integration by parts formula.

**Lemma 6.1.** *Suppose that one of $f,g\in\mathcal{D}_0$ has compact support in $(0,\infty)\times\mathbb{R}$. Then,*

$$\iint\nabla_a f\,d_tg\,dx=\iint\nabla_{-a}g^-\,d_tf\,dx. \tag{40}$$

*Proof.* Choosing any $a\in\mathbb{R}$, integration by parts and the condition that $fg$ has compact support in $(0,\infty)\times\mathbb{R}$ gives

$$\int f(t,x+a)\,d_tg(t,x)+\int g^-(t,x)\,d_tf(t,x+a)=0.$$

Then integrate with respect to $x$,

$$\begin{aligned}
&\iint f(t,x+a)\,d_tg(t,x)\,dx+\iint g^-(t,x-a)\,d_tf(t,x)\,dx=0.\\
&\iint f(t,x)\,d_tg(t,x)\,dx+\iint g^-(t,x)\,d_tf(t,x)\,dx=0.
\end{aligned}$$

The second of these equalities follows from the first by setting $a = 0$. Then, the result follows from subtracting the second of these equalities from the first and dividing by $a$. □

If we define the operator $D_a$ for $a \neq 0$ to be the difference of the left and right derivatives

$$D_a f \equiv \nabla_a f - \nabla_{-a} f$$

then the previous result can be used to prove the following limit.

**Corollary 6.2.** *Let $f, g \in \mathcal{D}_0$ and $\theta : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be measurable and bounded with bounded support. Then,*

$$\iint \theta D_a f \, d_t g \, dx + \iint \theta D_a g \, d_t f \, dx \to 0 \tag{41}$$

*as $a \to 0$.*

*Proof.* First, if we suppose that $f$ has compact support in $(0, \infty) \times \mathbb{R}$ then we can take the difference of equation (40) with itself with $a$ replaced by $-a$ to get

$$\iint D_a f \, d_t g \, dx + \iint D_a g^- \, d_t f \, dx = 0. \tag{42}$$

Now for general $f, g \in \mathcal{D}_0$ we choose any $\theta : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ that is continuously differentiable and has compact support in $(0, \infty) \times \mathbb{R}$. By replacing $f$ by $\theta f$ in equation (42),

$$\iint D_a(\theta f) \, d_t g \, dx + \iint \theta D_a g^- \, d_t f \, dx + \iint (D_a g^-) \frac{\partial \theta}{\partial t} f \, dt \, dx = 0 \tag{43}$$

For any function $h(t, x)$ which is locally Lipschitz continuous in $x$ we can make use of the identity $\nabla_{-a} h(t, x) = \nabla_a h(t, x - a)$ to get

$$\begin{aligned} \int u(x) D_a h(t, x) \, dx &= \int \left( u(x) \nabla_a h(t, x) - u(x) \nabla_a h(t, x - a) \right) dx \\ &= \int (u(x) - u(x + a)) \nabla_a h(t, x) \, dx \to 0 \end{aligned}$$

as $a \to 0$, whenever $u$ is continuous with compact support. In particular, we can combine this with dominated convergence for the integrals with respect to $t$ to get,

$$\begin{aligned} \iint (D_a g^-) \frac{\partial \theta}{\partial t} f \, dt \, dx &= \int \left( \int (D_a g^-) \frac{\partial \theta}{\partial t} f \, dx \right) dt \to 0, \\ \iint \theta (D_a (g^- - g)) d_t f \, dx &= \sum_{t \in \mathbb{R}_+} \int \theta D_a (g^- - g) \, (f - f^-) \, dx \to 0. \end{aligned}$$

as $a \to 0$. In the second of these limits we have used the fact that that there can only be countably many times at which $g^- \neq g$ in order to write the integral as an infinite sum. Combining these limits with equation (43) then gives

$$\iint D_a(\theta f)\, d_t g\, dx + \iint \theta D_a g\, d_t f\, dx \to 0$$

as $a \to 0$.

Now, the limit

$$\nabla_a(\theta f) - \theta \nabla_a f = f \nabla_a \theta + a \nabla_a \theta \nabla_a f \to f \frac{\partial \theta}{\partial x}$$

as $a \to 0$ implies that $D_a(\theta f) - \theta D_a f \to 0$. Applying this limit with dominated convergence to the first integral in equation (43) gives equation (41). The result for arbitrary bounded and measurable $\theta$ with bounded support now follows from the monotone class lemma. □

Finally for this section, we apply Corollary 6.2 to show that every $f \in \mathcal{D}$ is differentiable with respect to $x$ in an almost-everywhere sense. It is not clear whether or not this result will generalize to arbitrary $f \in \mathcal{D}_0$, which would then mean that we could remove the requirement for $f(t,x)$ to have left and right derivatives with respect to $x$ in Definition 2.6.

**Theorem 6.3.** *For any $f \in \mathcal{D}$ and $g \in \mathcal{D}_0$,*

$$\iint 1_{\{(t,x) \notin \operatorname{diff}(f)\}} |d_t g(t,x)|\, dx = 0.$$

*Proof.* Write $\nabla^l f(t,x)$ and $\nabla^r f(t,x)$ for the left and right derivatives respectively of $f(t,x)$ by $x$. Then, $f(t,x)$ is differentiable with respect to $x$ at those points where $\nabla^l f = \nabla^r f$.

From the definition of $D_a$ we have $D_a f \to \nabla^r f - \nabla^l f$ as $a \downarrow 0$. So equation (41) with $g$ replaced by $f$ gives

$$\iint \theta(\nabla^r f - \nabla^l f)\, d_t f\, dx = 0.$$

As this is true for every bounded and measurable $\theta$ with bounded support we have

$$\iint 1_{\{(t,x) \notin \operatorname{diff}(f)\}} |d_t f|\, dx = 0. \tag{44}$$

Similarly, equation (41) gives

$$\iint \theta(\nabla^r f - \nabla^l f)\, d_t g\, dx = -\lim_{a\downarrow 0} \iint \theta D_a g\, d_t f\, dx. \tag{45}$$

Now define the locally bounded function

$$K(t,x) \equiv \limsup_{a\downarrow 0} |D_a g(t,x)|$$

so that dominated convergence on the right hand side of equation (45) gives

$$\left| \iint \theta(\nabla^r f - \nabla^l f)\, d_t g\, dx \right| \leq \iint |\theta| K\, |d_t f|\, dx.$$

We can replace $\theta$ by $1_{\{(t,x)\notin \mathrm{diff}(f)\}}\theta$ in this inequality and apply equation (44),

$$\begin{aligned} \left| \iint \theta(\nabla^r f - \nabla^l f)\, d_t g\, dx \right| &\leq \iint 1_{\{(t,x)\notin \mathrm{diff}(f)\}} |\theta| K\, |d_t f|\, dx \\ &= 0. \end{aligned}$$

As this is true for every measurable and bounded $\theta$ with bounded support, the result follows. □

## 7 Proof of Theorem 2.7

The aim of this section is to apply Theorem 6.3 for the almost-everywhere differentiability of functions in $\mathcal{D}$ to the decomposition result of Theorem 5.2 in order to prove Theorem 2.7. We shall also give a proof of Theorem 2.5.

First, by stopping, we can reduce the semimartingale case to the case where $X$ decomposes as a sum of a martingale and an integrable variation process. Given any cadlag process $X$ and a stopping time $T$, we shall let $X^{T-}$ be the *pre-stopped* process

$$X_t^{T-} = \begin{cases} X_t, & \text{if } t < T, \\ X_{T-}, & \text{if } t \geq T > 0, \\ 0, & \text{if } T = 0. \end{cases}$$

Then we have the following simple result

**Lemma 7.1.** *Let $X$ be a semimartingale. Then, there exists a sequence of stopping times $T_n \uparrow \infty$ such that for each $n$, $X^{T_n-}$ has the following decomposition*

$$X^{T_n-} = M^n + A^n$$

*where $M^n$ is a cadlag martingale and $A^n$ is an integrable process with integrable variation.*

*Proof.* First, as $X$ is a semimartingale it has the decomposition

$$X = N + V$$

where $N$ is a local martingale with $N_0 = 0$ and $V$ has finite variation over each finite time interval. So, there exists stopping times $S_n \uparrow \infty$ such that

$$\mathbb{E}\left[\sup_{t\le S_n}|N_t|\right] < \infty$$

for each $n$. In particular, $N^{S_n}$ is a martingale. Let $T_n$ be the stopping time

$$T_n = \inf\left\{t \in \mathbb{R}_+ : 1_{\{t>0\}}|V_0| + \int_0^t |dV_s| \ge n \text{ or } t \ge S_n\right\}.$$

so that $V^{T_n-}$ is bounded by $n$ and also has variation bounded by $n$. It is also clear that $T_n \uparrow \infty$ as $n \to \infty$. We then have the decomposition

$$X_t^{T_n-} = N_t^{T_n} + V_t^{T_n-} - 1_{\{t\ge T_n>0\}}(\Delta N_{T_n}).$$

As $N^{T_n} = (N^{S_n})^{T_n}$ we see that it is a martingale. So, the result follows by setting $M^n = N^{T_n}$ and

$$A_t^n = V_t^{T_n-} - 1_{\{t\ge T_n>0\}}\Delta N_{T_n}.$$ □

Once we have a process $X$ that decomposes as a sum of a martingale and an integrable variation process, then we can represent its distribution at any time $t$ by taking the expectations of $(X_t - x)_+$ for each real $x$, which gives us an element of $\mathcal{D}$.

**Lemma 7.2.** *Let $X$ be an adapted cadlag process that decomposes as $X = M + A$ for a martingale $M$ and an integrable process $A$ with integrable variation over each finite time interval. Define the function $C : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ by*

$$C(t,x) \equiv \mathbb{E}\left[(X_t - x)_+\right].$$

*Then, $C(t,x)$ is convex in $x$, cadlag in $t$ and for every $x\in\mathbb{R}$*

$$C(t,x)+\mathbb{E}\left[\int_0^t|dA_s|\right]$$

*is increasing in $t$. In particular, $C\in\mathcal{D}$.*

*Proof.* First, $(X_t-x)$ is convex in $x$, so by the linearity of expectations $C(t,x)$ will also be convex in $x$. Also, from the decomposition of $X$ we see that $\{X_t:t\leq T\}$ is uniformly integrable for every $T>0$. Therefore, as $(X_t-x)_+$ is cadlag in $t$ we see that $C(t,x)$ will also be cadlag.

Let us now set $f(t)\equiv\mathbb{E}\left[\int_0^t|dA_s|\right]$, which is an increasing function. Then, for every $s<t$ we can use Jensen's inequality to get $\mathbb{E}\left[(M_t+A_s-x)_+\right]\geq\mathbb{E}\left[(M_s+A_s-x)_+\right]$, which gives

$$\begin{aligned}C(t,x)&=\mathbb{E}\left[(M_t+A_t-x)_+\right]\\&\geq\mathbb{E}\left[(M_t+A_s-x)_+\right]-\mathbb{E}\left[(A_t-A_s)_-\right]\\&\geq\mathbb{E}\left[(M_s+A_s-x)_+\right]-f(t)+f(s)\\&=C(s,x)-f(t)+f(s).\end{aligned}$$

So $C(t,x)+f(t)$ is increasing in $t$.

It only remains to show that $C\in\mathcal{D}$. First, the convexity in $x$ shows that $C(t,x)$ is locally Lipschitz continuous with left and right derivatives in $x$. Secondly, we can decompose $C(t,x)$ into a difference of increasing functions in $t$,

$$C(t,x)=(C(t,x)+f(t))-f(t)$$

so that its variation is bounded by

$$\int_0^T|d_tC(t,x)|\leq C(T,x)+2f(T).$$

So, for every $T>0$ and $K_0<K_1$

$$\begin{aligned}\int_{K_0}^{K_1}\int_0^T|d_tC(t,x)|\,dx&\leq\int_{K_0}^{K_1}\left(C(T,x)+2f(T)\right)\,dx\\&\leq(K_1-K_0)\left(C(T,K_0)+2f(T)\right)<\infty.\end{aligned}$$

□

We can now use this in order to simplify Theorem 5.2 in the case where $X$ is a semimartingale, by showing that equation (24) is always satisfied. Recall Definition 5.1 for the set of functions $\mathcal{D}_0$.

**Theorem 7.3.** *Let $X$ be a semimartingale and $f \in \mathcal{D}_0$. Suppose that*

$$\int 1_{\{(t,X_t)\notin \mathrm{diff}(f)\}}\, d[X]^c_t = 0. \tag{46}$$

*Then,*

$$f(t,X_t) = \int_0^t \nabla_+ f(s,X_{s-})\, dX_s + V_t \tag{47}$$

*where $V$ is a process with zero continuous quadratic variation.*

*Proof.* By Theorem 5.2, it is just necessary to show that equation (24) is satisfied. Also, by applying pre-localization Lemma 7.1 tells us that we can restrict to the case where $X$ has the decomposition $X = M + A$ for a martingale $M$ and an integrable process $A$ with integrable variation. We now set $C(t,x) \equiv \mathbb{E}\left[(X_t - x)_-\right]$. Then, letting $\nabla^l C$, $\nabla^r C$ be its left and right derivatives repectively in $x$, we have

$$\mathbb{P}\left(X_t = x\right) = \nabla^r C(t,x) - \nabla^l C(t,x).$$

Therefore,

$$\begin{aligned}
&\iint 1_{\{(t,x)\notin \mathrm{diffC}(f),\mathbb{P}(X_t=x)>0\}}\ |d_t f(t,x)|\ dx \\
&\le \iint 1_{\left\{\nabla^l C(t,x)\neq\nabla^r C(t,x)\right\}}\ |d_t f(t,x)|\ dx \\
&= \iint 1_{\{(t,x)\notin \mathrm{diff}(C)\}}\, |d_t f(t,x)|\, dx.
\end{aligned}$$

However, Lemma 7.2 says that $C \in \mathcal{D}$, so Theorem 6.3 tells us that the right hand side of the above inequality is 0. Therefore, equation (24) is satisfied and the result follows from Theorem 5.2. □

We shall use Theorem 7.3 to prove Theorems 2.7 and 2.5. However, we still need to show that equation (46) is satisfied. To this end, we shall make use of the following result relating the quadratic variation of $X$ to its marginal distributions and its drift and jump terms.

**Lemma 7.4.** *Let $X$ be a cadlag process which decomposes as $X = M + A$ for a martingale $M$ and cadlag integrable process $A$ with integrable variation over finite time intervals. Set $C(t,x) \equiv \mathbb{E}\left[(X_t - x)_+\right]$ so that, by Lemma 7.2, $C \in \mathcal{D}$.*

*Then, for any non-negative and measurable $\theta : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ with bounded support,*

$$\iint \theta\, d_t C\, dx = \frac{1}{2}\mathbb{E}\left[\int_0^\infty \theta(t, X_t)\, d[X]^c_t\right] + \mathbb{E}\left[\int_0^\infty \int_{-\infty}^{X_{t-}} \theta(t,y)\, dy\, dA_t\right] + \mathbb{E}\left[\sum_{t\in\mathbb{R}_+} \int_{X_{t-}}^{X_t} (X_t - x)\theta(t,x)\, dx\right]. \tag{48}$$

*Proof.* We only need to consider the case where $\theta$ is non-negative, twice continuously differentiable, and has compact support in $(0,\infty)\times\mathbb{R}$. The result would then follow by the monotone class lemma. So suppose that $\theta$ satisfies these properties and define $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ by

$$f(t,x) = \int \theta(t,y)(x-y)_+\, dy$$

so that $f$ is twice continuously differentiable with $f_{,22} = \theta$. Also, as $\theta$ has compact support, there exists $K, L \in \mathbb{R}_+$ such that $0 \le f(t,x) \le K(1+|x|)$ and $|f_{,2}| \le L$. Then Ito's formula (equations 1 and 2) says

$$f(t,X_t) = \int_0^t f_{,2}(s,X_{s-})\, dX_s + \frac{1}{2}\int_0^t \theta(s,X_s)\, d[X]^c_s + \int_0^t f_{,1}(s,X_s)\, ds + \sum_{s\in(0,t]} \int_{X_{s-}}^{X_s} (X_s - x)\theta(s,x)\, dx.$$

As $\int_0^t f_{,2}(s,X_{s-})\, dM_s$ is a local martingale, there exist stopping times $T_n \uparrow \infty$ such that

$$\mathbb{E}\left[\int_0^{t\wedge T_n} f_{,2}(s,X_{s-})\, dM_s\right] = 0$$

So,

$$\mathbb{E}\left[f(t\wedge T_n, X_{t\wedge T_n})\right] = \mathbb{E}\left[\int_0^{t\wedge T_n} f_{,2}(s,X_{s-})\, dA_s\right] + \frac{1}{2}\mathbb{E}\left[\int_0^{t\wedge T_n} \theta(s,X_s)\, d[X]^c_s\right] + \mathbb{E}\left[\int_0^{t\wedge T_n} f_{,1}(s,X_s)\, ds\right] + \mathbb{E}\left[\sum_{s\in(0,t\wedge T_n]} \int_{X_{s-}}^{X_s} (X_s - x)\theta(s,x)\, dx\right] \tag{49}$$

We can let $n$ go to infinity and use monotone convergence to show convergence of the second and fourth terms on the right hand side, and use

dominated convergence to prove convergence of the first and third terms. Also, uniform integrability of $X_{t\wedge T_n} = M_{t\wedge T_n} + A_{t\wedge T_n}$ over $n \in \mathbb{N}$ shows that the term on the left hand side will also converge.

So, if we choose $t$ large enough so that the support of $\theta$ is contained in $[0,t]\times\mathbb{R}$ then $f(t,X_t)=0$ and taking the limit as $n\to\infty$ in (49) gives

$$\begin{aligned}0 = {}&\mathbb{E}\left[\int f_{,2}(s,X_{s-})\,dA_s\right] + \frac{1}{2}\mathbb{E}\left[\int \theta(s,X_s)\,d[X]^c_s\right]\\ &+\mathbb{E}\left[\int f_{,1}(s,X_s)\,ds\right] + \mathbb{E}\left[\sum_{s>0}\int_{X_{s-}}^{X_s}(X_s-x)\theta(s,x)\,dx\right]\end{aligned}$$

The result now follows by substituting in

$$\mathbb{E}\left[\int f_{,2}(s,X_{s-})\,dA_s\right] = \mathbb{E}\left[\int\int_{-\infty}^{X_{s-}}\theta(s,y)\,dy\,dX_s\right]$$

and by using integration by parts

$$\begin{aligned}\int \mathbb{E}\left[f_{,1}(s,X_s)\right]\,ds &= \int \mathbb{E}\left[\int \theta_{,1}(s,x)(X_s-x)_+\,dx\right]\,ds\\ &= \iint \theta_{,1}(t,x)C(t,x)\,dt\,dx\\ &= -\iint \theta(t,x)\,d_tC(t,x)\,dx. \qquad \square\end{aligned}$$

A simple corollary of this is that equation (46) is satisfied whenever $f(t,x)$ is differentiable almost-everywhere in $x$ with respect to the measure $\iint\cdot\,|d_tC|\,dx$.

**Corollary 7.5.** *Let $X$ be a cadlag process that decomposes as $X=M+A$ for a martingale $M$ and cadlag integrable process $A$ with integrable variation over finite time intervals. Define $C\in\mathcal{D}$ by $C(t,x)\equiv\mathbb{E}\left[(X_t-x)_+\right]$.*

*If $f(t,x)$ is locally Lipschitz continuous in $x$ then,*

$$\mathbb{E}\left[\int 1_{\{(t,X_t)\notin\mathrm{diff}(f)\}}\,d[X]^c_t\right] \le 2\iint 1_{\{(t,x)\notin\mathrm{diff}(f)\}}\,|d_tC(t,x)|\,dx.$$

*Proof.* First, choose any non-negative bounded and measurable $\theta:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ with bounded support.

We shall use a result of Lebesgue which states that any locally Lipschitz continuous function from the reals to the reals is differentiable almost everywhere ([10] Theorem 3.2), giving

$$\int_{-\infty}^{X_{t-}} 1_{\{(t,y)\notin\mathrm{diff}(f)\}}\theta(t,y)\,dy = \int_{X_{t-}}^{X_t} 1_{\{(t,x)\notin\mathrm{diff}(f)\}}(X_t-x)\theta(t,x)\,dx = 0.$$

So, replacing $\theta(t,x)$ by $1_{\{(t,x)\notin\mathrm{diff}(f)\}}\theta(t,x)$ in equation (48) gives

$$\begin{aligned}&\mathbb{E}\left[\int 1_{\{(t,X_t)\notin\mathrm{diff}(f)\}}\theta(t,X_t)\,d[X]^c_t\right]\\&=2\iint 1_{\{(t,x)\notin\mathrm{diff}(f)\}}\,\theta(t,x)d_tC(t,x)\,dx\\&\leq 2\iint 1_{\{(t,x)\notin\mathrm{diff}(f)\}}\,\theta(t,x)|d_tC(t,x)|\,dx\end{aligned}$$

Letting $\theta$ increase to 1 gives the result. □

We have now done enough to complete the proof of Theorem 2.7.

*Proof of Theorem 2.7.* First, by Lemma 7.1, we may restrict to the case where $X$ is a sum of a martingale and a cadlag integrable process with integrable variation over finite time intervals. Then Corollary 7.5 and Theorem 6.3 give

$$\mathbb{E}\left[\int 1_{\{(t,X_t)\notin\mathrm{diff}(f)\}}\,d[X]^c_t\right]\leq 2\iint 1_{\{(t,x)\notin\mathrm{diff}(f)\}}\,|d_tC(t,x)|\,dx=0.$$

So, the result follows from Theorem 7.3. □

In a similar way, we can now finally complete the proof of Theorem 2.5. In this case we do not use Theorem 6.3, but instead make use of Lebesgue's result that locally Lipschitz continuous functions on the reals are differentiable almost everywhere ([10] Theorem 3.2).

*Proof of Theorem 2.5.* By Lemma 7.1, we may restrict to the case where $X$ is a sum of a martingale and a cadlag integrable process with integrable variation over finite time intervals. We then define $C\in\mathcal{D}$ by $C(t,x)=\mathbb{E}\left[(X_t-x)_+\right]$.

Let $\mathrm{diff}(f)$ be the set of $x\in\mathbb{R}$ at which $f$ is differentiable. Also, choose any $t>0$ and set $\theta(x)=\int_0^t|d_sC(s,x)|$. Corollary 7.5 gives

$$\begin{aligned}\mathbb{E}\left[\int_0^t 1_{\{X_s\notin\mathrm{diff}(f)\}}\,d[X]^c_s\right]&\leq 2\int_{-\infty}^{\infty}\int_0^t 1_{\{x\notin\mathrm{diff}(f)\}}|d_sC(s,x)|\,dx\\&=2\int 1_{\{x\notin\mathrm{diff}(f)\}}\theta(x)\,dx.\end{aligned}$$

However, as $f$ is locally Lipschitz continuous, Lebesgue's theorem tells us that $f$ is differentiable almost everywhere, and the right hand side of the above equality is 0. In particular, equation (46) is satisfied and the result follows from Theorem 7.3. □

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