arXiv:0801.0912v2 [math.AC] 14 Feb 2008

DIOPHANTINE INEQUALITY FOR EXCELLENT HENSELIAN LOCAL DOMAINS

HIROTADA ITO, SHUZO IZUMI

ABSTRACT. G. Rond has proved a Diophantine type inequality for the field of quotients of the convergent or formal power series ring in multivariables. We generalize his theorem to the field of the quotients of an excellent Henselian local domain whose reduced order function is expressed as the minimum of a finite set of good valuations in Rees's sense.

RÉSUMÉ. G. Rond a démontré une inégalité de type Diophantien pour le corps des quotients de séries convergentes (ou formelles) à plusieurs variables. On fait ici une généralisation de son théorème au corps des quotients d'un anneau local intégral Henselien excellent dont la fonction d'ordre réduit s'exprime comme le minimum d'un ensemble de nombre fini de bonnes valuations au sens de Rees.

2000 Mathematics Subject Classification Number: 13H10, 11J25

KEY WORDS: Diophantine inequality for a local ring,
Linear Artin approximation property, m-valuation.

1. INTRODUCTION

An important topic of Diophantine approximation is the problem of approximation of a real algebraic number by rational ones. The crucial result is Roth's theorem:

If $z \in \mathbb{R} \setminus \mathbb{Q}$ is an algebraic number,

$$\forall \epsilon > 0\ \exists c(z,\ \epsilon) > 0\ \forall x \in \mathbb{Q}\ \forall y \in \mathbb{Q}^* :\ \left|z - \frac{x}{y}\right| > c(z,\ \epsilon)|y|^{-2+\epsilon}.$$

There are quite similar results for the Laurent series field in a single variable (cf. [L]). It is also known that there are deep analogous results on the global function fields on certain special varieties in connection with Nevanlinna's theory (cf. [Ru]).

Rond [Ro2] obtained a Diophantine inequality for the field of quotients of the convergent or formal power series ring in multivariables in connection with the *linear Artin approximation property* (Spivakovsky, cf. [Ro1]). He used the *product inequality* (Izumi's theorem [I1]) for the order function ν on an analytic domain.

In this paper we assert that Rond's inequality holds for the field of quotients of an excellent Henselian local domain whose reduced order function is expressed as the minimum of a finite set of good valuations in Rees's sense. For the proof, we use *Rees's inequality* [Re3] *for* $\mathfrak{m}$*-valuations*, which is a strengthened form of the product inequality. An inequality related to the product one was once used to measure transcendence of elements of a local ring in [I2]. This time we apply its variant to elements algebraic over a local ring.

Date: June 14, 2019.

Let us give a precise description of our theorem. Let K be a (commutative) field. We call a mapping $\boldsymbol{v} : K \longrightarrow \overline{\mathbb{R}}$ ($\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$) a *valuation* when it satisfies the following:

(1) $\boldsymbol{v}(xy) = \boldsymbol{v}(x) + \boldsymbol{v}(y)$,
(2) $\boldsymbol{v}(x + y) \geq \min\{\boldsymbol{v}(x),\ \boldsymbol{v}(y)\}$,
(3) $\boldsymbol{v}(0) = \infty$.

We put:

$V_{\boldsymbol{v}} := \{z \in K : \boldsymbol{v}(z) \geq 0\}$: the *valuation ring* of $\boldsymbol{v}$,
$\mathfrak{m}_{\boldsymbol{v}} := \{z \in K : \boldsymbol{v}(z) > 0\}$: the *maximal ideal* of $V_{\boldsymbol{v}}$,
$k_{\boldsymbol{v}} := V_{\boldsymbol{v}}/\mathfrak{m}_{\boldsymbol{v}}$: the *residue field* of $\boldsymbol{v}$,
$\hat{K}$: the completion of K with respect to $\boldsymbol{v}$,
$\hat{\boldsymbol{v}}$: the continuous extension of $\boldsymbol{v}$ to $\hat{K}$ with respect to the discrete topology on $\mathbb{R}$,
$\hat{V}_{\boldsymbol{v}}$: the valuation ring of the extension $\hat{\boldsymbol{v}}$.

Let $(A,\ \mathfrak{m})$ be a local domain and let $K := Q(A)$ denote its field of quotients. We define the $\mathfrak{m}$-*adic order* $\nu_{\mathfrak{m}} : A \longrightarrow \overline{\mathbb{R}}$ on A by $\nu_{\mathfrak{m}}(f) := \max\{p : f \in \mathfrak{m}^p\}$. This is not necessarily a valuation. It satisfies formulae

(1′) $\nu_{\mathfrak{m}}(fg) \geq \nu_{\mathfrak{m}}(f) + \nu_{\mathfrak{m}}(g)$,
(2) $\nu_{\mathfrak{m}}(f + g) \geq \min\{\nu_{\mathfrak{m}}(f),\ \nu_{\mathfrak{m}}(g)\}$,
(3′) $\nu_{\mathfrak{m}}(0) = \infty$, $\nu_{\mathfrak{m}}(1) = 0$.

Let us stabilize $\nu_{\mathfrak{m}}$ by Samuel's idea: $\overline{\nu}_{\mathfrak{m}}(f) := \lim_{k\to\infty} \nu_{\mathfrak{m}}(f^k)/k$. This limit always exists and satisfies formulae (1′), (2), (3′) and the homogeneity formula

(4) $\overline{\nu}_{\mathfrak{m}}(f^d) = d\overline{\nu}_{\mathfrak{m}}(f)$

also (see [Re1]). Then our main result is the following.

Suppose that $\overline{\nu}_{\mathfrak{m}}$ is expressed as the minimum of a finite set of $\mathfrak{m}$-valuations (a good valuation defined in §2) and that A is excellent and Henselian. If $z \in \hat{K} \setminus K$ is algebraic over K, then we have the following:

$$\exists a \geq 0\ \exists c > 0\ \forall x \in A\ \forall y \in A^* :\ \left|z - \frac{x}{y}\right|_{\hat{\boldsymbol{v}}} > c|y|_{\boldsymbol{v}}^{a}.$$

Here $A^* := A \setminus \{0\}$ and the absolute values $|\ |_{\boldsymbol{v}} : K \longrightarrow \mathbb{R}$, $|\ |_{\hat{\boldsymbol{v}}} : \hat{K} \longrightarrow \mathbb{R}$ are defined as

$$|w|_{\boldsymbol{v}} := \exp(-\boldsymbol{v}(w)),\ |w|_{\hat{\boldsymbol{v}}} := \exp(-\hat{\boldsymbol{v}}(w)).$$

The essential point of the proof is Rond's idea in [Ro2] which reduces inequality on valuation to inequality on order (see §4, (v)).

In the contrary to the case of algebraic numbers, the exponent on the right of this inequality is not uniformly bounded. Rond ([Ro1], **2.4**) has given a sequence of elements $z_i \in \hat{K}$ of degree 2 over K with unbounded exponents.

2. Order and $\mathfrak{m}$-valuations

Let $(A,\ \mathfrak{m})$ be a local domain whose field of quotients $Q(A)$ is K. Let $k := A/\mathfrak{m}$ denote the residue field. A valuation $\boldsymbol{v}$ on K is called an $\mathfrak{m}$-*valuation*, if it satisfies the following:

(a) $x \in A \Longrightarrow \boldsymbol{v}(x) \geq 0$,
(b) $x \in \mathfrak{m} \Longrightarrow \boldsymbol{v}(x) > 0$,
(c) $\operatorname{trdeg}_k k_{\boldsymbol{v}} = \dim A - 1$,
(d) The value group $\boldsymbol{v}(K^*)$ is isomorphic to $\mathbb{Z}$ (as an ordered group).

Let us recall key facts on valuations which is used in the proof. The first one is Rees's *strong valuation theorem* [Re2]. We state only the special case which we need later.

Fact 2.1. *Let* $(A,\ \mathfrak{m})$ *be a Noetherian local ring whose completion has no non-zero nilpotent element and let* $\nu_{\mathfrak{m}}$ *denote the* $\mathfrak{m}$*-adic order. Then there exist a non-negative number* C *and a set of* $\mathbb{Q}$*-valued valuations* $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ *on* K *such that*

$$\forall x \in A:\ \nu_{\mathfrak{m}}(x) \leq \overline{\nu}_{\mathfrak{m}}(x) \leq \nu_{\mathfrak{m}}(x) + C,\ \forall x \in A:\ \overline{\nu}_{\mathfrak{m}}(x) = \min\{\boldsymbol{v}_1(x), \ldots, \boldsymbol{v}_p(x)\}.$$

The set $\{\boldsymbol{v}_1(x), \ldots, \boldsymbol{v}_p(x)\}$ *is unique, if it is taken irredundant.*

We call $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ the *valuations associated with* $\mathfrak{m}$. (This definition is slightly different from Rees's.) We call a local ring *analytically irreducible* when its completion is an integral domain. Another fact is Rees's inequality [Re3], (E) which is a more general and stronger form of the product inequality [I1]:

Fact 2.2. *Let* $(A,\ \mathfrak{m})$ *be an analytically irreducible Noetherian local ring. Then for any pair of* $\mathfrak{m}$*-valuations* $\boldsymbol{v}$ *and* $\boldsymbol{v}'$*, we have the following.*

$$\exists d > 0\ \forall x \in A:\ \boldsymbol{v}(x) \leq d\boldsymbol{v}'(x).$$

Combining these facts we see the following:

Fact 2.3. *Let* $(A,\ \mathfrak{m})$ *be an analytically irreducible Noetherian local ring. Suppose that the valuations associated with* $\mathfrak{m}$ *are all* $\mathfrak{m}$*-valuations and let* $\boldsymbol{v}$ *be an* $\mathfrak{m}$*-valuation. Then we have:*

$$\exists C > 0\ \exists s > 0\ \exists t > 0\ \forall x \in A:\ s\boldsymbol{v}(x) \leq \overline{\nu}_{\mathfrak{m}}(x) \leq \nu_{\mathfrak{m}}(x) + C \leq t\boldsymbol{v}(x) + C.$$

3. Main theorem

Let $(A,\ \mathfrak{m})$ be a local domain with field of quotients $K := Q(A)$ and let $\boldsymbol{v} : K \longrightarrow \overline{\mathbb{R}}$ ($\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$) be an $\mathfrak{m}$-valuation. The valuation ring $V_{\boldsymbol{v}} := \{x \in K : \boldsymbol{v}(x) \geq 0\}$ is a discrete valuation ring (DVR) with maximal ideal $\mathfrak{m}_{\boldsymbol{v}} := \{x \in K : \boldsymbol{v}(x) > 0\}$, $\boldsymbol{v}$ coincides with the $\mathfrak{m}_{\boldsymbol{v}}$-adic order on $V_{\boldsymbol{v}}$ and we have $K = Q(A) = Q(V_{\boldsymbol{v}})$. It is well-known that $V_{\boldsymbol{v}}$ is a principal ideal domain and it is integrally closed. The absolute value $|x|_{\boldsymbol{v}} := \exp(-\boldsymbol{v}(x))$ of $x \in K$ defines a translation invariant distance on K endowing it a topology compatible with the field operations. Then we can define the completion $\hat{K}$ of K and extend $\boldsymbol{v}$ to a valuation $\hat{\boldsymbol{v}}$ on $\hat{K}$ naturally. The completion $\hat{B}$ of some subset $B \subset K$ can be identified with its closure. The completion $\hat{V}_{\boldsymbol{v}}$ can be identified with the valuation ring of $\hat{\boldsymbol{v}}$ also. With these notations we have the following:

Theorem 3.1. *Let* $(A,\ \mathfrak{m})$ *be an excellent Henselian local integral domain and let* $K := Q(A)$ *denote its field of quotients and let* $\boldsymbol{v} : K \longrightarrow \overline{\mathbb{R}}$ *be an* $\mathfrak{m}$*-valuation. Suppose that* A *satisfies the following condition:*

($*$) *The valuations associated with* $\mathfrak{m}$ *are all* $\mathfrak{m}$*-valuations.*

If $z \in \hat{K} \setminus K$ *is algebraic over* K*. Then we have*

$$\exists a \geq 0\ \exists c > 0\ \forall x \in A\ \forall y \in A^*:\ \left|z - \frac{x}{y}\right|_{\hat{\boldsymbol{v}}} > c|y|_{\boldsymbol{v}}^a.$$

In view of the inequality in the last part of §2, we may replace $|y|_{\boldsymbol{v}}^a = \exp(-\boldsymbol{v}(y))$ by $\exp(-\nu_{\mathfrak{m}}(y))$ (with a modified a).

Remark 3.2. A ring of germs of analytic functions on a reduced and irreducible complex analytic singularities satisfy all the conditions for Theorem. The condition ($*$) follows from the following facts. The order is expressed by the valuations defined by irreducible components of exceptional divisor of the point blowing-up followed by a Hironaka resolution (cf. [LT]) and the function fields on theses components are projective and have transcendence

degrees equal to $\dim A - 1$, which proves the condition (c) in §2. According to Rees [Re3], all the complete local integral domains also satisfy all the conditions for Theorem.

Just in the same way as Rond [Ro2], **3.1** (see [Ro1], **2.1** also), our Theorem **3.1** implies the following.

Corollary 3.3. *Let* $(A,\ \mathfrak{m})$ *be an excellent Henselian integral domain which satisfies the condition* $(*)$ *and let* $P(X,\ Y) \in A[X,\ Y]$ *be a homogeneous polynomial. Then the Artin function of* $P(X,\ Y)$ *is majorised by an affine function i.e.*

$$\exists\alpha\ \exists\beta\ \forall x \in A\ \forall y \in A :\ \nu_{\mathfrak{m}}(P(x,\ y)) \geq \alpha i + \beta$$

$$\Longrightarrow \exists\overline{x} \in A\ \exists\overline{y} \in A :\ \nu_{\mathfrak{m}}(\overline{x} - x) \geq i,\ \nu_{\mathfrak{m}}(\overline{x} - x) \geq i,\ P(\overline{x},\ \overline{y}) = 0.$$

This corollary reminds us of the theorem that an excellent Henselian local ring has the strong Artin approximation property (cf. [P]).

4. Proof of Theorem

(i) Reduction to normal case.

We may assume that $\boldsymbol{v}(K^*) = \boldsymbol{v}(\hat{K}^*) = \mathbb{Z}$. This results in a change of the exponent a. Let $\tilde{A}$ denote the normalization (the integral closure in K) of A. Since A is a Henselian integral domain, $\tilde{A}$ is a local ring by [N], **43.11**, **43.20**. Since A is excellent,

(1) A is a G-ring and a Nagata (= pseudo-geometric) ring

by [M], **33.H**. Then $\tilde{A}$ is a finite A-module. Hence $\dim A = \dim \tilde{A}$ by a theorem of Cohen-Seidenberg (cf. [N], **10.10**) and $r\tilde{A} \subset A$ for some $r \in A^*$ (existence of a universal denominator). Then a Diophantine inequality for $\tilde{A}$ implies one for A with the same exponent a. Finiteness also implies that $\tilde{A}$ is excellent and Henselian by [N], **43.16**.

Let $\tilde{\mathfrak{m}}$ denote the maximal ideal of $\tilde{A}$. We claim that $\boldsymbol{v}$ is an $\tilde{m}$-valuation. If $x \in \tilde{A}$,

$$\exists p \in \mathbb{N},\ \exists b_0, \ldots, b_{p-1} \in A :\ x^p = b_o + b_1 x + \cdots + b_{p-1}x^{p-1}.$$

Then we have $p\boldsymbol{v}(x) \geq \min\{i\boldsymbol{v}(x) : 0 \leq i \leq p-1\}$. This proves $\tilde{A} \subset V_{\boldsymbol{v}}$ and condition (a) for $(\tilde{A},\ \tilde{\mathfrak{m}})$. Let us put $\overline{\mathfrak{m}} := \{x \in \tilde{A} : \boldsymbol{v}(x) > 0\}$. Then $\overline{\mathfrak{m}}$ is a prime ideal of $\tilde{A}$ and $\overline{\mathfrak{m}} \cap A = \mathfrak{m}$. This implies that $\overline{\mathfrak{m}} = \tilde{\mathfrak{m}}$ by [B], Chapt.5, 2.1, Prop. 1 and (b) holds. Since $\tilde{A}$ is a finite A-module, $\tilde{k} = \tilde{A}/\tilde{\mathfrak{m}}$ is a finite k-module ($k := A/\mathfrak{m}$) i.e. $\tilde{k}$ is algebraic over k. This proves (c). The condition (d) is obvious. We have proved the claim and we may assume that

(2) A is an excellent, Henselian and normal local domain.

(ii) Reduction of the minimal equation.

Let

$$\varphi(Z) := a_0 + a_1 Z + \cdots + a_d Z^d\ (a_d \neq 0,\ d \geq 2)$$

be a minimal equation for z over A, that is, φ is a polynomial of the minimal degree in $A[Z]$ with $\varphi(z) = 0$. Now take $u \in A^*$ and put

$$\varphi_u(Z) := u^d a_d^{d-1} \varphi(Z/ua_d).$$

Then we have

$$\varphi_u(Z) = a_0 u^d a_d^{d-1} + a_1 u^{d-1} a_d^{d-2} Z + \cdots + Z^d \in A[Z].$$

Then $w' \in \hat{K}$ is a root of $\varphi_u(Z)$ if and only if $w := w'/ua_d$ is a root of $\varphi(Z)$. If

$$\exists a \geq 0\ \exists c > 0\ \forall x \in A\ \forall y \in A :\ \left|z' - \frac{x}{y}\right|_{\hat{v}} > c|y|_v^a$$

holds for $z' := ua_d z$, we have

$$\exists a \geq 0\ \exists c > 0\ \forall x \in A\ \forall y \in A:\ \left|z - \frac{x}{y}\right|_{\hat{v}} > \frac{c}{|ua_d|}|y|_v^a.$$

The polynomial $\varphi_u(Z) \in A[Z]$ is minimal for z'. Thus, choosing u, we may assume that $z \in \hat{V}_v$ and

$$\varphi(Z) := a_0 + a_1 Z + \cdots + a_{d-1}Z^{d-1} + Z^d \ (d \geq 2,\ a_i \in \mathfrak{m}^{d-i})$$

from the first.

(iii) Order function on $A[z]$, the ring generated by z over A.

Let us consider the residue ring $B := A[Z]/\varphi(Z)A[Z]$. There is an isomorphism $\iota : B \longrightarrow A[z]$. The ring B is a finite A-module with basis $1, z^2, \ldots, z^{d-1}$. Since $A[z]$ is a subring of the field $\hat{K}$, B is an integral domain. Thus we have the following:

(3) $\qquad A[z] \cong B := A[Z]/\varphi(Z)A[Z]$ is an integral extension of A.

Since A is Henselian, B is a local ring by (3) and [N], **(43.12)**. As a consequence of (ii), $z^d \in \mathfrak{m}A[z]$. Hence the maximal ideal of B is $\mathfrak{n} := \mathfrak{m}B + ZB$ and its residue ring is the same as that of A: $k = A/\mathfrak{m} = B/\mathfrak{n}$. Let us define $\mu : A[Z] \longrightarrow \overline{\mathbb{R}}$ by

$$\mu(\sum_{i=0}^{e} b_i Z^i) := \min_i\{\nu_{\mathfrak{m}}(b_i) + i\}\ (b_i \in A)$$

and $\nu_{\mathfrak{n}} : B \longrightarrow \overline{\mathbb{R}}$ as the $\mathfrak{n}$-adic order. The function μ is nothing but the restriction of the standard order on the formal power series ring $A[[Z]]$. We claim that $\nu_{\mathfrak{n}}(x)$ coincides with the μ-order of the unique representative of x in $A[Z]$ of degree less than d i.e.

$$\mu(\sum_{i=0}^{d-1} b_i Z^i) = \nu_{\mathfrak{n}}(\sum_{i=0}^{d-1} b_i Z^i \mod \varphi(Z)A[Z]).$$

We have only to show that inequality

$$\mu(\sum_{i=0}^{d-1} b_i Z^i) < \mu(\sum_{i=0}^{d-1} b_i Z^i + \sum_{0}^{d} a_i Z^i \sum_{j=0}^{e} c_j Z^j),$$

leads us to a contradiction. Let us develop the product $\sum_{i=0}^{d} a_i Z^i \sum_{i=0}^{e} c_j Z^j$ and reduce its degree in Z by repeated substitutions $Z^d = -\sum_{i=0}^{d-1} a_i Z^i$, beginning from the highest degree term. By the assumption $a_i \in \mathfrak{m}_A^{d-i}$, the substitutions do not lower the μ-order and we reach the left side. This contradicts the inequality we assumed.

The function $\nu_{\mathfrak{n}}$ induces $\nu := \nu_{\mathfrak{n}} \circ \iota^{-1} : A[z] \longrightarrow \overline{\mathbb{R}}$. Of course ν inherits the *non-cancellation property* from $\nu_{\mathfrak{n}}$:

$$\nu(\sum_{i=0}^{d-1} b_i z^i) = \min\{\nu(b_i) + i : 0 \leq i \leq d-1\} = \min\{\nu_{\mathfrak{m}}(b_i) + i : 0 \leq i \leq d-1\}.$$

In other words, there occurs no cancellation among summands of degree less than d.

(iv) $A[z]$ is analytically irreducible.

It seems that the analytical irreducibility of $A[z]$ trivially follows from the assertion that the closure of $A[z]$ is a subring of the field $\hat{K}$. This is not a correct argument because we do not know at present that the completion of $A[z]$ with respect to its maximal ideal is isomorphic to the closure of $A[z]$ in $\hat{K}$ (whose topology is defined by $\hat{v}$). To prove this isomorphism we need to show that $A[z]$ is analytically irreducible, a circular argument (see below).

Since A is a normal G-ring,

(4) A is analytically normal, i.e. its completion is normal,

by (2), (3) and [M], **33.I**.

By (1), (3), (4), we see that $A[z]$ is analytically irreducible by [N], **44.1**.

Let $\mathfrak{m}'$ denote the maximal ideal of $A[z]$. We claim that the restriction $\hat{\boldsymbol{v}}|Q(A[z])$ is an $\mathfrak{m}'$-valuation. By the assumptions that $z \in \hat{V}_{\boldsymbol{v}}$ in (ii) and that $\mathfrak{m}'$ is generated by $\mathfrak{m}$ and z, the conditions (a), (b) are satisfied. Since $A[z]$ is a finite A-module, $\dim A[z] = \dim A$ by a theorem of Cohen-Seidenberg. Since $A[z]/\mathfrak{m}'$ is a finite k-module and $Q(A[z]) \cong K[z]$, we have only to prove $\operatorname{trdeg}_k k_{\boldsymbol{v}} = \operatorname{trdeg}_k k'$ to prove (c), where $k' := (\hat{V}_{\boldsymbol{v}} \cap K[z])/(\hat{\mathfrak{m}}_{\boldsymbol{v}} \cap k[z])$. Since $\boldsymbol{v}(K^*) = \hat{\boldsymbol{v}}(\hat{K}^*) = \mathbb{Z}$, there exists $x \in K^*$ such that $\boldsymbol{v}(x) = \hat{\boldsymbol{v}}(z)$. Then any elements of k' is represented by $\sum_{i=0}^{d-1} y_i(z/x)^i$ ($y_i \in K$, $\boldsymbol{v}(y_i) = 0$), which proves that k' is a finite $k_{\boldsymbol{v}}$-module and their transcendence degrees agree. The conditions (d) is trivial.

Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_p$ be the valuations on $Q(A[z])$ associated with $\mathfrak{m}'$ (see (**2.1**)). Then we have equality

$$\overline{\nu}_{\mathfrak{m}}(x) = \overline{\nu}_{\mathfrak{n}}(x) = \min\{\boldsymbol{v}_1|_K(x), \dots, \boldsymbol{v}_p|_K(x)\}$$

on K by (iii). Irredundancy of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_p$ implies that of $\boldsymbol{v}_1|_K, \dots, \boldsymbol{v}_p|_K$. Thus each of $\boldsymbol{v}_1|_K, \dots, \boldsymbol{v}_p|_K$ are the valuations associated with $\mathfrak{m}$ and defines the same completion $\hat{K}$ of K by (**2.3**). Then, in the same way as $\hat{\boldsymbol{v}}$ (except that $\boldsymbol{v}_i(K^*) = \hat{\boldsymbol{v}}_i(\hat{K}^*)$ may not be $\mathbb{Z}$), we see that $\boldsymbol{v}_1, \dots, \boldsymbol{v}_p$ are $\mathfrak{m}'$-valuations. This proves that $A[z]$ has the property $(*)$. Since the completion of $A[z]$ is an integral domain as a subring of $\hat{K}$, we can apply (**2.3**) to $A[z]$.

(v) Diophantine inequality.

If $\hat{\boldsymbol{v}}(z - \dfrac{x}{y}) \le \hat{\boldsymbol{v}}(z)$, we have $\left|z - \frac{x}{y}\right|_{\hat{\boldsymbol{v}}} \ge \exp(-\hat{\boldsymbol{v}}(z))$ at once. Hence we may assume that $\hat{\boldsymbol{v}}(x - yz) - \boldsymbol{v}(y) > \hat{\boldsymbol{v}}(z)$.

If $\boldsymbol{v}(x) \ne \hat{\boldsymbol{v}}(yz)$, we have a contradiction: $\hat{\boldsymbol{v}}(z) < \hat{\boldsymbol{v}}(x - yz) - \boldsymbol{v}(y) \le \hat{\boldsymbol{v}}(yz) - \boldsymbol{v}(y) = \hat{\boldsymbol{v}}(z)$. Hence we have only to consider the case

$$\boldsymbol{v}(x) = \hat{\boldsymbol{v}}(yz) = \boldsymbol{v}(y) + \hat{\boldsymbol{v}}(z).$$

Since $A[z]$ is analytically irreducible, applying the inequality (**2.3**) and the equality at the last part of (iii), we have

$$\hat{\boldsymbol{v}}(x - yz) \le \frac{1}{s}(\nu(x - yz) + C) \le \frac{1}{s}(\nu(x) + C) \le \frac{t}{s}\boldsymbol{v}(x) + \frac{C}{s}.$$

It follows that

$$\hat{\boldsymbol{v}}(z - \frac{x}{y}) \le \frac{t}{s}\boldsymbol{v}(x) - \boldsymbol{v}(y) + \frac{C}{s} = \frac{t - s}{s}\boldsymbol{v}(y) + \frac{t}{s}\hat{\boldsymbol{v}}(z) + \frac{C}{s}.$$

This implies the inequality of our theorem.

References

[B] N. Bourbaki: Éléments de mathématique, Algébre commutative, Hermann, Paris 1965

[I1] S. Izumi: A measure of integrity for local analytic algebra, Publ. RIMS Kyoto Univ. **21**, 719–735 (1985)

[I2] S. Izumi: Transcendence measures for subsets of local algebras, in: Real analytic and algebraic singularities (ed. T. Fukuda et al.), Pitman Res. Notes Math. 381, Longman, Edinburgh Gate 1998, 189–206

[L] A. Lasjaunias: A survey of Diophantine approximation in fields of power series, Monatsh. Math. 130, 211–229 (2000)

[LT] M. Lejeune-Jalabert, T. Teissier: Clöture integral des idéaux et équisingularite, Univ. Sc. et Medicale de Grenoble, 1974.

[M] H. Matsumura: Commutative algebra, second edition, Benjamin/Commings, MLNS 56, London 1980

[N] M. Nagata: Local rings, Interscience, ITPAM 13, New York 1962

[P] D. Popescu: Artin approximation, in Handbook of algebra vol. 2 (ed. by Hazelwinkel), North-Holland, Amsterdam 2000, 321–356

[Re1] D. Rees: Valuation associated with a local ring (I), Proc. London Math. Soc. (3), **5**, 107–128 (1955)

[Re2] D. Rees: Valuations associated with ideals II, J. London Math. Soc. (3) **31**, 221–228 (1956)

[Re3] D. Rees: Izumi's theorem, in: Commutative algebra (ed. by Hochster et al.), Math. Sci. Res. Inst. Pub., **15**, 407–416, Springer (1989)

[Ro1] G. Rond: Sur la liéarité de la foncton de Artin, Ann. Scient. Éc. Norm. Sup. **38**, 979–988 (2005)

[Ro2] G. Rond: Approximation diophantienne dans les corps de séries en plusieurs variables, Ann. Institut Fourier, **56**-2, 299–308 (2006)

[Ru] M. Ru: Nevanlinna theory and its relation to Diophantine approximation, World Scientific, Singapore 2001