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Stable Tameness of Two-Dimensional Polynomial Automorphisms Over a Dedekind Domain

Joost Berson and David Wright

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Abstract

In this paper it is established that all two-dimensional polynomial automorphisms over a Dedekind $\mathbb{Q}$-algebra are stably tame; in fact, they become tame with the addition of three more dimensions. A key element in the proof is this additional new theorem: Over an Artinian $\mathbb{Q}$-algebra all two-dimensional polynomial automorphisms having Jacobian determinant one are tame.

1 Introduction

The famous theorem of Jung and van der Kulk ([4],[5]) asserts that all two-dimensional polynomial automorphisms over a field are tame. (See §2 for the definition of tameness and other terminology.) Jung proved this for fields of characteristic zero and van der Kulk generalized it to arbitrary characteristic. It is well-known that this fails to be true over a domain R which is not a field. A standard example of a non-tame automorphism is

$$\big(X + a(aY + X^2), Y - 2X(aY + X^2) + a(aY + X^2)^2\big)$$

where a is any non-zero non-unit in R. For $R = k[T]$ and $a = T$, k a field, this is the famous example of Nagata [7] which he conjectured to be non-tame as a 3-dimensional automorphism over k. Shestakov and Umirbaev [10] finally proved Nagata's conjecture. Meanwhile it had been shown by Smith [8] and Wright (unpublished) that Nagata's example is stably tame, in fact tame with the addition of one more variable[1]. The matter of stable tameness is one of intrigue because no example has been produced (to the authors' knowledge) of a polynomial automorphism over a domain which cannot be shown to be stable tame.

The main result of this paper is Theorem 4.3, which asserts that all two-dimensional polynomial automorphisms over a Dedekind domain become tame with the addition of three new dimensions. It is proved by a somewhat delicate argument for which Theorem 4.1 serves as the enabling preparation lemma. The latter result, which says that all two-dimensional automorphisms over an Artinian $\mathbb{Q}$-algebra are tame, can be viewed as a generalization of Jung's Theorem [4]. Also used in the proof of the main theorem are four technical lemmas and a result of Suslin, all of which appear in §3.

[1] Smith's method uses the fact that this automorphism is the exponential of a locally nilpotent derivation. However, one can modify Nagata's example slightly so that is does not appear to be such an exponential, but still becomes tame with one new dimension.

2 Notation and Terminology

We use the following fairly standard notations for the polynomial automorphism groups over a commutative ring R. We write $R^{[n]}$ for the polynomial ring $R[X_1, \ldots, X_n]$. $\mathrm{GA}_n(R)$ denotes the automorphism group of $\operatorname{Spec} R$. As such, it is anti-isomorphic to the group of R-algebra automorphisms of $R^{[n]}$. An element of $\mathrm{GA}_n(R)$ is represented by a vector $\varphi = (F_1, \ldots, F_n) \in (R^{[n]})^n$, and we often write X for the identity element $(X_1, \ldots, X_n)$ of $\mathrm{GA}_n(R)$. We write $J\varphi$ for the Jacobian matrix of an automorphism φ. The general linear group $\mathrm{GL}_n(R)$ is contained in $GA_n(R)$ in an obvious way. We have the following other subgroups (and we here suppress R):

- SA_n is the subgroup of all φ for which $|J\varphi| = 1$. (Here and throughout this paper $|\ |$ denotes determinant.)
- EA_n is the subgroup generated by the elementary automorphisms. An *elementary* automorphism is one of the form

  $$(X_1, \ldots, X_{i-1}, X_i + f(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n), X_{i+1}, \ldots, X_n)$$

  for some $i \in \{1, \ldots, n\}$. An elementary automorphism of the above form for a specific i is called *elementary in the ith position.*
- T_n, the group of *tame* automorphisms, is the subgroup generated by GL_n and EA_n.
- Tr_n is the subgroup of translations. A *translation* is an automorphism of the form $(X_1 + a_1, \ldots, X_n + a_n)$ with $a_1, \ldots, a_n \in R$. This group is isomorphic to the additive group R^n.
- Af_n, the *affine* group, is the subgroup generated by GL_n and Tr_n. It is, in fact, the semidirect product $\mathrm{GL}_n \ltimes \mathrm{Tr}_n$, with GL_n acting by conjugation on $\mathrm{Tr}_n \cong R^n$ in the obvious way.

We will use standard notation for the linear groups such as SL_n, E_n (the subgroup of GL_n generated by elementary matrices), D_n (the group of invertible diagonal matrices), and GE_n (the subgroup generated by E_n and D_n).

If G and H are subgroups of some group, we write $\langle G, H\rangle$ for the subgroup generated by $G \cup H$. For example $\mathrm{T}_n = \langle \mathrm{GL}_n, \mathrm{EA}_n\rangle$ and $\mathrm{GE}_n = \langle \mathrm{D}_n, \mathrm{E}_n\rangle$.

Given t a non-zero-divisor in a ring R, we write R_t for the localization $R[1/t]$. For $a \in R_t$ and n an integer ≥ 0 we say "a has t-order $\geq -n$" if $t^n a \in R$. Also if γ is a matrix over R_t we say "γ has t-order $\geq -n$" if $t^n\gamma$ has entries in R.

We say $\varphi, \psi \in \mathrm{GA}_n(R)$ are *tamely equivalent* (respectively *elementarily equivalent*) if there exist ϵ, ϵ' in $\mathrm{T}_n(R)$ (resp. $\mathrm{EA}_n(R)$) such that $\epsilon\varphi\epsilon' = \psi$. To show that an automorphism is tame (resp. a product of elementaries) we may replace it by an automorphism to which it is tamely (resp. elementarily) equivalent.

The results herein use the concept of *stabilization,* which refers to the embedding of $\mathrm{GA}_n(R)$ into $\mathrm{GA}_{n+m}(R)$ (the "stabilization homomorphism"). If $\varphi \in \mathrm{GA}_n(R)$, we write $\varphi^{[m]}$ for its image in $\varphi \in \mathrm{GA}_{n+m}(R)$; we also sometimes just write φ for $\varphi^{[m]}$. We say, for example, an automorphism φ is *stably tame* if it becomes tame in some higher dimension. We sometimes specify the number of dimensions by saying "φ becomes tame with the addition of m dimensions (or variables)", meaning $\varphi^{[m]}$ is tame.

We will also encounter the "restriction of scalars" embedding, by which we view $\mathrm{GA}_m(R^{[n]})$ as the subgroup of $\mathrm{GA}_{n+m}(R)$ which fixes (anti-isomorphically) the first n variables. By this identification we have $\mathrm{EA}_m(R^{[n]}) \subset \mathrm{EA}_{n+m}(R)$, but the embedding does <u>not</u> necessarily place $\mathrm{T}_m(R^{[n]})$ within $\mathrm{T}_{n+m}(R)$. In fact elements of $\mathrm{GL}_m(R^{[n]})$ may not be tame over R. This fact presents an obstruction in the proof of the main theorem, Theorem 4.3, which requires the use of Theorem 3.5 to overcome.

3 Preliminaries

In order to prove Theorem 4.1, we need the two following lemmas. The statement of the first appears in [1], § 5.2, as Exercise 7.

Lemma 3.1. *Let R be a $\mathbb{Q}$-algebra and X and Y two variables. Then every monomial X^nY^m in the polynomial ring $R^{[2]} = R[X,Y]$ can be written as a $\mathbb{Q}$-linear combination of polynomials of the form $(X+aY)^{n+m}$, with $a \in \mathbb{Q}$.*

Proof. For every $k \in \{0,\ldots,n+m\}$, we have the identity

$$(X+kY)^{n+m} = \sum_{i=0}^{n+m} \tbinom{n+m}{i} k^i X^{n+m-i} Y^i$$

Now define vectors $v, w \in R[X,Y]^{n+m+1}$ by

$$v = \left(X^{n+m}, (X+Y)^{n+m}, (X+2Y)^{n+m}, \ldots, (X+(n+m)Y)^{n+m}\right)$$
$$w = \left(\tbinom{n+m}{0}X^{n+m}, \tbinom{n+m}{1}X^{n+m-1}Y, \ldots, \tbinom{n+m}{n+m-1}XY^{n+m-1}, \tbinom{n+m}{n+m}Y^{n+m}\right)$$

Then $v = \alpha w$, where the square matrix $\alpha = (a_{ij})$ is given by $a_{ij} = (i-1)^{j-1}$. Hence, α is a Vandermonde matrix, which implies that its determinant is an element of $\mathbb{Q}^*$. The inverse of α, together with the inverse of $\binom{n+m}{n}$, now give the desired expression for X^nY^m. □

The following lemma borrows ideas from [2].

Lemma 3.2. *Let $\mathfrak{a}$ be an ideal in a commutative ring R, satisfying $\mathfrak{a}^2 = (0)$. Suppose $F, G \in \mathfrak{a}\left(R^{[n]}\right)^n$, and define $\phi, \gamma \in GA_n(R)$ by $\phi = X + F$, $\gamma = X + G$ (note, that ϕ and γ are indeed invertible: $\phi^{-1} = X - F$, and $\gamma^{-1} = X - G$). Then $\phi\gamma = X + F + G$.*

Proof. Straightforward. □

Now we present two technical lemmas on elementary equivalence which will be needed in the proof of the main theorem (Theorem 4.3).

Lemma 3.3. *Let R be a ring and $t \in R$ a non-zero-divisor. The map*

$$(X + F(X,Y), Y + G(X,Y)) \mapsto (Z + \frac{1}{t}F(X+tZ, Y+tW), W + \frac{1}{t}G(X+tZ, Y+tW))$$

defines a group homomorphism $\Psi_t : GA_2(R) \to GA_2(R_t[X,Y])$. Moreover, if t divides F and G in $R[X,Y]$, and if $\varphi = (X+F, Y+G) \in GA_2(R)$, then $\Psi_t(\varphi)$ lies in $GA_2(R[X,Y])$ and is elementarily equivalent to $\varphi^{[2]}$ in $GA_4(R)$.

Proof. Letting $\eta = (X, Y, Z + (1/t)X, W + (1/t)Y), \sigma = (X - tZ, Y - tW, Z, W) \in \mathrm{GA}_4(R_t)$, a direct computation shows that $\Psi_t(\varphi) = \sigma\eta\varphi^{[2]}\eta^{-1}\sigma^{-1}$. This shows that Ψ_t is a group homomorphism. However, we can also write $\Psi_t(\varphi) = \sigma\varphi^{[2]}\omega\sigma^{-1}$ where $\omega = (X, Y, Z + (1/t)F(X,Y), W + (1/t)G(X,Y))$. If t divides F and G then ω is elementary over R, and since $\sigma \in \mathrm{EA}_4(R)$ it follows that $\Psi_t(\varphi)$ is elementarily equivalent over R to $\varphi^{[2]}$. □

Lemma 3.4. *Let R be a ring and $t \in R$ a non-zero-divisor. Let $n \geq 0$ be an integer and let $\alpha \in SL_2(R_t)$ having t-order $\geq -n$. Let $\varepsilon = (X, Y + f(X)) \in GA_2(R)$ with $f(X) \in t^{n+dn}R[X]$, where $d = \deg f(X)$. Then $\alpha^{-1}\varepsilon\alpha \in GA_2(R)$ and it lies in $EA_3(R)$.*

Proof. Writing $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have $\alpha^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then

$$\begin{aligned} \alpha^{-1}\varepsilon\alpha &= (dX - bY, -cX + aY) \circ (X, Y + f(X)) \circ (aX + bY, cX + dY) \\ &= (X - bf(aX + bY), Y + af(aX + bY)) \end{aligned}$$

Writing $f(X) = t^{n+dn}g(X)$ with $g(X) \in R[X]$, we see that we can further write $f(X) = t^n h(t^n X)$, with $h(X) \in R[X]$. Writing

$$t^n\alpha = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$$

with $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in R$, we have $\alpha^{-1}\varepsilon\alpha = (X - \tilde{b}h(\tilde{a}X + \tilde{b}Y), Y + \tilde{a}h(\tilde{a}X + \tilde{b}Y)$. Adding one more dimension, represented by the variable Z, $(\alpha^{-1}\varepsilon\alpha)^{[1]}$ becomes the commutator $\kappa\nu\kappa^{-1}\nu^{-1}$ where $\kappa = (X - \tilde{b}Z, Y + \tilde{a}Z, Z)$, $\nu = (X, Y, Z + h(\tilde{a}X + \tilde{b}Y))$. Note that $\kappa, \nu \in \mathrm{EA}_3(R)$. □

Finally, we will also employ the following result of Suslin, which is a special case of [9], Corollary 6.5.

Theorem 3.5 (Suslin)**.** *Let R be a Dedekind domain. Then*

$$GL_m\left(R^{[n]}\right) = \left\langle E_m\left(R^{[n]}\right), GL_m(R)\right\rangle$$

for $m \geq 3$.

4 The Main Results

The following theorem will be an important component in the proof of Theorem 4.3, but it is also of interest in its own right. Again, some ideas from [2] are employed.

Theorem 4.1. *Let R be an Artinian $\mathbb{Q}$-algebra. Then every automorphism φ in two variables over R with $|J\varphi| = 1$ is a composition of elementary automorphisms. In other words, $SA_2(R) = EA_2(R)$.*

Proof. Since R is Artinian it is a product of Artinian local rings ([6], Ch. X, Theorem 7.7). Since GA_n splits over a product, we may assume from now on, that R is an Artinian local ring.

So now we have a local ring $(R, \mathfrak{m})$ with $\mathfrak{m}^N = 0$ for some integer $N \geq 1$. We will prove by induction on N that F is a product of elementary automorphisms.

The case $N = 1$ follows from the famous Jung-van der Kulk Theorem. So now let $N \geq 2$ and let $\bar{R} = R/\mathfrak{m}^{N-1}$. Since $\bar{\varphi} \in \mathrm{GA}_2(\bar{R})$ with $\det J\bar{\varphi} = 1$, $\bar{\varphi}$ is a composition of elementary automorphisms, i.e., $\varphi \in \mathrm{EA}(\bar{R})$. We can lift each of these elementary automorphisms to elementary automorphisms over R to produce $\varepsilon \in \mathrm{EA}_2(R)$ such that $\varepsilon^{-1}\varphi = (X + f, Y + g)$, where $f, g \in \mathfrak{m}^{N-1}R[X, Y]$. Note, that also $|J(\varepsilon^{-1}\varphi)| = 1$. So we may assume, that $\varphi = (X + f, Y + g)$. Since $\mathfrak{m}^N = (0)$, $|J(\varphi)| = 1 + \frac{\partial f}{\partial X} + \frac{\partial g}{\partial Y}$. Then $\frac{\partial f}{\partial X} + \frac{\partial g}{\partial Y} = 0$, and since R is a $\mathbb{Q}$-algebra, this implies that there exists a polynomial $p \in R[X, Y]$ such that $f = \frac{\partial p}{\partial Y}$ and $g = -\frac{\partial p}{\partial X}$. Using Lemma 3.2, we may assume that $p = rX^nY^m$ for some $r \in R$, $n, m \geq 0$. (Note: If $\deg(p) \leq 1$, then f and g are constants and we are done.) With Lemma 3.1, we can write X^nY^m as a $\mathbb{Q}$-linear combination of polynomials of the form $(X + aY)^{n+m}$, with $a \in \mathbb{Q}$. Using Lemma 3.2 again, we may assume that

$$\varphi = (X + kabr(X + aY)^{k-1}, Y - kbr(X + aY)^{k-1}),$$

where $k \geq 1$, $a, b \in \mathbb{Q}$ and $r \in R$. But then $\varphi = \alpha^{-1}\beta\alpha$, where $\alpha = (X + aY, Y)$ and $\beta = (X, Y - kbrX^{k-1})$. □

Remark 4.2. We do not know whether Theorem 4.1 holds for an Artinian ring R which is not a $\mathbb{Q}$-algebra, except in the case where R is a finite product of local rings each of which is either a $\mathbb{Q}$-algebra or a field of positive characteristic. In this case the conclusion follows from Theorem 4.1 together with van der Kulk's Theorem.

We now prove:

Theorem 4.3 (Main Theorem)**.** *Let R be a $\mathbb{Q}$-algebra which is a Dedekind domain, and let $\varphi \in GA_2(R)$. Then φ is stably tame. More strongly, φ becomes tame with the addition of three more dimensions. In other words $GA_2(R) \subset T_5(R)$.*

Proof. Let $\varphi \in \mathrm{GA}_2(R)$. We may assume $|J(\varphi)| = 1$. Let K be the field of fractions of R. Then $\mathrm{GA}_2(K) = \mathrm{T}_2(K)$ by Jung's Theorem. Since $\mathrm{T}_2(K) = \langle \mathrm{GL}_2(K), \mathrm{EA}_2(K)\rangle$, we can write φ in the form $\varphi = \alpha_1\varepsilon_1 \cdots \alpha_r\varepsilon_r$, where $\alpha_1, \ldots, \alpha_r \in \mathrm{GL}_2(K)$ and, for $i = 1, \ldots, r$, $\varepsilon_i = (X, Y + f_i(X))$, with $f_i \in K[X]$. (To get all the ε_i to be elementary in the second position we use the formula $(Y, X) \circ (X + g(Y), Y) \circ (Y, X) = (X, Y + g(X)$.) Moreover, we can arrange that $\alpha_1, \ldots, \alpha_r$ lie in $\mathrm{SL}_2(K)$; this is accomplished by sweeping diagonal elements to the left and using the fact that $|J(\varphi)| = 1$. Let $d_i = \max(\deg f_i, 1)$.[2]

Choose an element $t \in R$ such that each $f_i(X)$ and all coefficients of each α_i lie in $R_t[X]$. We will now carefully choose an integer $N \geq 0$ such that t^N will be a sufficiently large power of t to enable some repeated applications of Lemmas 3.3 and 3.4. For $i = 1, \ldots, r$, let $n_i, \ell_i \geq 0$ be such that $f_i(X)$ has t-order $\geq -n_i$ and α_i has t-order $\geq -\ell_i$. Let N_i be defined by $N_r = n_r + \ell_r$ and $N_j = n_j + \ell_j + d_jN_{j+1}$ for $j = 1, \ldots, r-1$. Secondly, define the sequence $M_1, \ldots, M_r$ by $M_r = N_r$ and $M_j = n_j + d_jN_{j+1} + (d_j + 1)(N_{j+1} + N_{j+2} + \cdots + N_r)$. Finally choose an integer $N \geq \max(N_1, M_1, M_2, \ldots, M_r)$. Note that $N \geq N_j$ for each j since $N_j \geq N_{j+1}$.

Let $\bar{R} = R/t^NR$. We consider the image $\bar{\varphi}$ of φ in $\mathrm{GA}_2(\bar{R})$, which must also have the property $|J(\bar{\varphi})| = 1$. Since $\bar{R}$ is Artinian (R being a Dedekind domain), Theorem 4.1 gives that $\bar{\varphi} \in \mathrm{EA}_2(\bar{R})$. Write $\bar{\varphi}^{-1} = \bar{\rho}_1 \cdots \bar{\rho}_s$, a product of elementary automorphisms. Each $\bar{\rho}_j$ lifts to an elementary automorphism ρ_j in $\mathrm{GA}_2(R)$. We can replace φ by $\varphi\rho_1 \cdots \rho_s$, so that φ has the factorization

$$\varphi = \alpha_1\varepsilon_1 \cdots \alpha_r\varepsilon_r\rho_1 \cdots \rho_s\,, \tag{1}$$

and $\bar{\varphi} = (\bar{X}, \bar{Y})$. We can now write $\varphi = (X + t^NF, Y + t^NG)$ with $F, G \in R[X, Y]$.

Now we apply the map $\Psi = \Psi_{t^N}$ of Lemma 3.3, and note that

$$\begin{aligned}\Psi(\varphi) &= \Big(X, Y, Z + F(X + t^NZ, Y + t^NW)\,,\, W + G(X + t^NZ, Y + t^NW)\Big)\\ &= \Psi(\alpha_1)\Psi(\varepsilon_1) \cdots \Psi(\alpha_r)\Psi(\varepsilon_r)\Psi(\rho_1) \cdots \Psi(\rho_s)\end{aligned} \tag{2}$$

lies in $\mathrm{GA}_4(R)$, and that, by Lemma 3.3, φ is elementarily equivalent to $\Psi(\varphi)$ in $\mathrm{GA}_4(R)$. For the moment we will view $\Psi(\varphi)$ as an element of $\mathrm{GA}_2(R[X, Y])$ and the factors of (2) as elements of $\mathrm{GA}_2(K[X, Y])$, by restriction of scalars. We now go through a process of "moving translations and linear parts to the left", the result of which will show $\Psi(\varphi) \in \langle \mathrm{EA}_3(R[X, Y]), \mathrm{GL}_5(R)\rangle$ (via the stabilization embedding $\mathrm{EA}_3(R[X, Y]) \subset \mathrm{GA}_5(R)$) and hence $\Psi(\varphi)$ is tame as an element of $\mathrm{GA}_5(R)$.

First we consider $\Psi(\rho_s)$. If $\rho_s = (X, Y + r_s(X))$ with $r(X) \in R[X]$, then $\Psi(\rho_s)$, viewed as an element of $\mathrm{GA}_2(K[X, Y])$, has the form $\Psi(\rho_s) = (Z, W + (1/t^N)r_s(X + t^NZ))$. By Taylor's expansion we have $(1/t^N)r_s(X + t^NZ) = (1/t^N)r_s(X) + \tilde{r}_s(Z)$ with $\tilde{r}_s(Z) \in R[X, Z]$. Thus we have $\Psi(\rho_s) = (Z, W + (1/t^N)r_s(X)) \circ \tilde{\rho}_s$ where $\tilde{\rho}_s = (Z, W + \tilde{r}_s(Z))$; note that $\tilde{\rho}_s$ is elementary over $R[X, Y]$ (and over R). Now we consider the product $\Psi(\rho_{s-1})\Psi(\rho_s)$. Writing $\rho_{s-1} = (X + r_{s-1}(Y), Y)$ (We may assume the elementary automorphisms $\rho_1, \ldots, \rho_s$

[2] Actually, if $\deg f_i = 0$ (or if $f_i = 0$), we could shorten the factorization of φ, but for the sake of simplicity (for example, to be able to write the factorization of φ so as to end with ε_r rather than α_r) we will allow this possibility.

alternate in this fashion, otherwise we could shorten the product.), then we have $\Psi(\rho_{s-1}) = (Z + (1/t^N)r_{s-1}(Y + t^N W), W)$. Hence

$$\begin{aligned}
\Psi(\rho_{s-1})\Psi(\rho_s) &= \left(Z + \frac{1}{t^N}r_{s-1}(Y + t^N W), W\right) \circ \left(Z, W + \frac{1}{t^N}r_s(X)\right) \circ \tilde{\rho}_s \\
&= \left(Z + \frac{1}{t^N}r_{s-1}(Y + r_s(X) + t^N W), W + \frac{1}{t^N}r_s(X)\right) \circ \tilde{\rho}_s \\
&= \left(Z + \frac{1}{t^N}r_{s-1}(Y + r_s(X)) + \tilde{r}_{s-1}(X, Y, W), W + \frac{1}{t^N}r_s(X)\right) \circ \tilde{\rho}_s \\
&\qquad\qquad \text{with } \tilde{r}_{s-1}(X, Y, W) \in R[X, Y, W] \\
&= \left(Z + \frac{1}{t^N}r_{s-1}(Y + r_s(X)), W + \frac{1}{t^N}r_s(X)\right) \circ (Z + \tilde{r}_{s-1}(X, Y, W), W) \circ \tilde{\rho}_s \\
&= \left(Z + \frac{1}{t^N}r_{s-1}(Y + r_s(X)), W + \frac{1}{t^N}r_s(X)\right) \circ \tilde{\rho}_{s-1}\tilde{\rho}_s
\end{aligned}$$

where $\tilde{\rho}_{s-1} = (Z + \tilde{r}_{s-1}(X, Y, W), W))$, which is again elementary over $R[X, Y]$. We continue this process of sweeping translations to the left to get

$$\Psi(\rho_1) \cdots \Psi(\rho_s) = \tau\tilde{\rho}_1 \cdots \tilde{\rho}_s$$

with $\tilde{\rho}_1, \ldots, \tilde{\rho}_s$ elementary over $R[X, Y]$ and τ a translation over $K[X, Y]$ of the form

$$\tau = \left(Z + \frac{1}{t^N}p(X, Y), W + \frac{1}{t^N}q(X, Y)\right) \tag{3}$$

with $p(X, Y), q(X, Y) \in R[X, Y]$. We now have $\Psi(\varphi) = \Psi(\alpha_1)\Psi(\varepsilon_1) \cdots \Psi(\alpha_r)\Psi(\varepsilon_r)\tau\tilde{\rho}_1 \cdots \tilde{\rho}_s$. Since our goal is to show $\Psi(\varphi)$ is tame in $\mathrm{GA}_5(R)$ and since each $\tilde{\rho}_j$ is elementary over R, we may replace $\Psi(\varphi)$ by

$$\psi = \Psi(\alpha_1)\Psi(\varepsilon_1) \cdots \Psi(\alpha_r)\Psi(\varepsilon_r)\tau \tag{4}$$

and it becomes our goal to show that ψ is tame in $\mathrm{GA}_5(R)$.

Now we examine $\Psi(\varepsilon_r)\tau$,which appears at the end of the factorization (4). Since $\varepsilon_r = (X, Y + f_r(X))$ we have $\Psi(\varepsilon_r) = (Z, W + (1/t^N)f_r(X + t^N Z))$ and hence, from (3):

$$\begin{aligned}
\Psi(\varepsilon_r)\tau &= \left(Z, W + \frac{1}{t^N}f_r(X + t^N Z)\right) \circ \left(Z + \frac{1}{t^N}p(X, Y), W + \frac{1}{t^N}q(X, Y)\right) \\
&= \left(Z + \frac{1}{t^N}p(X, Y), W + \frac{1}{t^N}q(X, Y) + \frac{1}{t^N}f_r(X + p(X, Y) + t^N Z)\right) \\
&= \left(Z + \frac{1}{t^N}p(X, Y), W + \frac{1}{t^N}\left[q(X, Y) + f_r(X + p(X, Y))\right] + f_r'(X + p(X, Y))Z + \tilde{f}_r(X, Y, Z)\right) \\
&= \left(Z + \frac{1}{t^N}p(X, Y), W + \frac{1}{t^N}\left[q(X, Y) + f_r(X + p(X, Y))\right]\right) \circ \left(Z, W + f_r'(X + p(X, Y))Z\right) \\
&\qquad \circ \left(Z, W + \tilde{f}_r(X, Y, Z)\right)
\end{aligned} \tag{5}$$

Since $N \geq n_r$ we have $\tilde{f}_r(X, Y, Z) \in R[X, Y, Z]$. Let us write the final factorization (5) as $\Psi(\varepsilon_r)\tau = \tau_r'\beta_r\tilde{\varepsilon}_r$ and record that $\tau_r' = (Z + (1/t^N)u_r(X, Y), W + (1/t^N)v_r(X, Y))$ with $u_r, v_r \in R_t[X, Y]$ both having t-order $\geq -n_r$, that $\beta_r \in \mathrm{SL}_2(R_t[X, Y])$ having t-order $\geq -n_r$, and that $\tilde{\varepsilon}_r$ is elementary over R. We now have

$$\psi = \Psi(\alpha_1)\Psi(\varepsilon_1) \cdots \Psi(\alpha_{r-1})\Psi(\varepsilon_{r-1})\Psi(\alpha_r)\tau_r'\beta_r\tilde{\varepsilon}_r$$

In order to sweep τ_r' past $\Psi(\alpha_r)$, we make this observation: Given any $\alpha \in \mathrm{SL}_2(R_t)$, viewed as an element of $\mathrm{GA}_2(R_t)$, write $\alpha = (X + aX + bY, Y + cX + dY)$, with $a, b, c, d \in R_t$, and note that

$$\Psi(\alpha) = \left(Z + aZ + bW + \frac{1}{t^N}(aX + bY), W + cZ + dW + \frac{1}{t^N}(cX + dY) \right).$$

Letting $\alpha = \alpha_r$ one can easily see that $\Psi(\alpha_r)\tau_r'$ can be written as $\tau_r\hat{\alpha}_r$ with $\hat{\alpha}_r = (Z + aZ + bW, W + cZ + dW) \in \mathrm{SL}_2(R_t[X, Y])$ (with coefficients in R_t, even) having t-order $\geq -\ell_r$, and τ_r a translation over $R_t[X, Y]$. More precisely, we can write $\tau_r = (Z + (1/t^N)p(X, Y), W + (1/t^N)q(X, Y))$ with p, q having order $\geq -(n_r + \ell_r)$. Finally, let $\gamma_r = \hat{\alpha}_r\beta_r$ and note that $\gamma_r \in \mathrm{SL}_2(R_t[X, Y])$ has t-order $\geq -(n_r + \ell_r)$. Recall that $N_r = n_r + \ell_r$.

We now have $\psi = \Psi(\alpha_1)\Psi(\varepsilon_1)\cdots\Psi(\alpha_{r-1})\Psi(\varepsilon_{r-1})\tau_r\gamma_r\tilde{\varepsilon}_r$, and since $\tilde{\varepsilon}_r$ is elementary over R we see that ψ is elementarily equivalent in $\mathrm{GA}_4(R)$ to

$$\psi_r = \Psi(\alpha_1)\Psi(\varepsilon_1)\cdots\Psi(\alpha_{r-1})\Psi(\varepsilon_{r-1})\tau_r\gamma_r$$

Now, inductively, suppose $2 \leq j \leq r$ and suppose we have shown that ψ is elementarily equivalent in $\mathrm{GA}_5(R)$ (the need for dimension 5 will arise in the induction and in the final step of the proof) to

$$\psi_j = \Psi(\alpha_1)\Psi(\varepsilon_1)\cdots\Psi(\alpha_{j-1})\Psi(\varepsilon_{j-1})\tau_j\gamma_j$$

with $\tau_j = (Z + (1/t^N)p_j(X, Y), W + (1/t^N)q_j(X, Y))$, p_j, q_j having t-order $\geq -N_j$, and $\gamma_j \in \mathrm{SL}_2(R_t[X, Y])$ having t-order $\geq -(N_j + N_{j+1} + \cdots + N_r)$. We will show that we can achieve this same situation with j replaced by $j - 1$.

We have $\Psi(\varepsilon_{j-1}) = (Z, W + (1/t^N)f_{j-1}(X + t^N Z))$, so as in (5) above we obtain

$$\begin{aligned}
\Psi(\varepsilon_{j-1})\tau_j &= \left(Z, W + \frac{1}{t^N}f_{j-1}(X + t^N Z) \right) \circ \left(Z + \frac{1}{t^N}p_j(X, Y), W + \frac{1}{t^N}q_j(X, Y) \right) \\
&= \left(Z + \frac{1}{t^N}p_j(X, Y), W + \frac{1}{t^N}\left[q_j(X, Y) + f_{j-1}(X + p_j(X, Y)) \right] \right) \\
&\qquad \circ \left(Z, W + f_{j-1}'(X + p_j(X, Y))Z \right) \circ \left(Z, W + \tilde{f}_{j-1}(X, Y, Z) \right) \qquad (6)
\end{aligned}$$

and we write this composition (6) as $\Psi(\varepsilon_{j-1})\tau_j = \tau_{j-1}'\beta_{j-1}\tilde{\varepsilon}_{j-1}$. Noting that $q_j(X, Y) + f_{j-1}(X + p_j(X, Y))$ has t-order $\geq -(n_{j-1} + d_{j-1}N_j)$ we can conclude that τ_{j-1}' possesses the form $(Z + (1/t^N)u_{j-1}(X, Y), W + (1/t^N)v_{j-1}(X, Y))$ where $u_{j-1}, v_{j-1} \in R_t[X, Y]$ with order $\geq -(n_{j-1} + d_{j-1}N_j)$. Observe that $f_{j-1}'(X + p_j(X, Y))$ also has t-order $\geq -(n_{j-1} + d_{j-1}N_j)$, hence $\beta_{j-1} \in \mathrm{SL}_2(R_t[X, Y])$ having t-order $\geq -(n_{j-1} + d_{j-1}N_j)$. Finally note that $\tilde{f}_{j-1}(X, Y, Z) \in R[X, Y, Z]$ where it is divisible by $t^{N - n_{j-1} - d_{j-1}N_j}$.

Moreover, we can sweep τ_{j-1}' past $\Psi(\alpha_{j-1})$ as before to get $\Psi(\alpha_{j-1})\tau_{j-1}' = \tau_{j-1}\hat{\alpha}_{j-1}$ with $\hat{\alpha}_{j-1} \in \mathrm{SL}_2(R_t[X, Y])$ having t-order $\geq -\ell_{j-1}$ and $\tau_{j-1} = (Z + (1/t^N)p_{j-1}(X, Y), W + (1/t^N)q_{j-1}(X, Y))$ with p_{j-1}, q_{j-1} having order $\geq -(n_{j-1} + \ell_{j-1} + d_{j-1}N_j) = -N_{j-1}$. We have

$$\begin{aligned}
\psi_j &= \Psi(\alpha_1)\Psi(\varepsilon_1)\cdots\Psi(\alpha_{j-2})\Psi(\varepsilon_{j-2})\tau_{j-1}\hat{\alpha}_{j-1}\beta_{j-1}\tilde{\varepsilon}_{j-1}\gamma_j \\
&= \Psi(\alpha_1)\Psi(\varepsilon_1)\cdots\Psi(\alpha_{j-2})\Psi(\varepsilon_{j-2})\tau_{j-1}\hat{\alpha}_{j-1}\beta_{j-1}\gamma_j\gamma_j^{-1}\tilde{\varepsilon}_{j-1}\gamma_j \\
&= \Psi(\alpha_1)\Psi(\varepsilon_1)\cdots\Psi(\alpha_{j-2})\Psi(\varepsilon_{j-2})\tau_{j-1}\gamma_{j-1}\gamma_j^{-1}\tilde{\varepsilon}_{j-1}\gamma_j
\end{aligned}$$

where $\gamma_{j-1} = \hat{\alpha}_{j-1}\beta_{j-1}\gamma_j$, which has t-order $\geq -(n_{j-1} + \ell_{j-1} + d_{j-1}N_j + N_j + \cdots + N_r) = -(N_{j-1} + N_j + \cdots + N_r)$. Now, $\tilde{\varepsilon}_{j-1} = (Z, W + \tilde{f}_{j-1}(X, Y, Z))$ with $\tilde{f}_{j-1}$ divisible by $t^{N - n_{j-1} - d_{j-1}N_j}$. Now we wish to apply Lemma 3.4 to see that $\gamma_j^{-1}\tilde{\varepsilon}_{j-1}\gamma_j$ lies in $\mathrm{EA}_3(R[X, Y])$.

Since γ_j has t-order $\geq -(N_j + \cdots + N_r)$, this will be the case provided $N - n_{j-1} - d_{j-1}N_j \geq (d_{j-1}+1)(N_j + \cdots + N_r)$, which is just the statement $N \geq M_{j-1}$. Since $\mathrm{EA}_3(R[X,Y]) \subset \mathrm{E}_5(R)$ we see that ψ_j is elementarily equivalent in $\mathrm{GA}_5(R)$ to

$$\psi_{j-1} = \Psi(\alpha_1)\Psi(\varepsilon_1)\cdots\Psi(\alpha_{j-2})\Psi(\varepsilon_{j-2})\tau_{j-1}\gamma_{j-1}$$

with τ_{j-1}, γ_{j-1} of the desired form.

We continue to get ψ elementarily equivalent in $\mathrm{GA}_5(R)$ to $\psi_1 = \tau_1\gamma_1 \in \mathrm{Af}_2(R_t[X,Y]) \cap \mathrm{GA}_2(R[X,Y]) = \mathrm{Af}_2(R[X,Y])$. In particular, τ_1 is a translation over $R[X,Y]$, so ψ is tamely equivalent to $\gamma_1 \in \mathrm{GL}_2(R[X,Y])$. According to Theorem 3.5, $\gamma_1^{[1]}$ lies in $\langle \mathrm{E}_3(R[X,Y]), \mathrm{GL}_3(R)\rangle$, which is contained in $T_5(R)$. This completes the proof of Theorem 4.3. □

Remark 4.4. If R is a Euclidean domain we have $\mathrm{SL}_m(R) = \mathrm{E}_m(R)$ for all $m \geq 1$. In this case we can conclude (since $\mathrm{E}_n(R) \subset \mathrm{EA}_n(R)$) that $\varphi \in \langle \mathrm{EA}_5(R), \mathrm{D}_5(R)\rangle$, which equals $\mathrm{T}_5(R)$. (Caution: Not all Dedekind domains – in fact, not all PIDs – have this property. See [3].) Taking $R = k[T]$, k a field of characteristic zero, we get $\mathrm{GA}_2(k[T]) \subset \mathrm{T}_6(k)$ by virtue of the restriction of scalars and stabilization embeddings.

Remark 4.5. We do not know whether three additional dimensions are needed to achieve tameness over a Dedekind domain. As was pointed out in the introduction, the example of Nagata, and other similar examples, become tame with just one more dimension.

Remark 4.6. The only place where we used that R is a $\mathbb{Q}$-algebra is in the appeal to Theorem 4.1. As with Theorem 4.1 (see Remark 4.2), we do not know whether Theorem 4.3 remains valid without this assumption.

We conclude with the following, which derives immediately from Theorem 4.3, together with Remark 4.4 and the fact that the polynomial ring $k[T]$ is a Euclidean domain.

Corollary 4.7. *Let k be a field of characteristic zero and let W be the subgroup of $GA_3(k)$ generated by all automorphisms which fix one coordinate. Then all elements of W are stably tame. More precisely, $W \subset T_6(k) = \langle EA_6(k), D_6(k)\rangle$.*

Note that W properly contains the tame subgroup $\mathrm{T}_3(k)$, as Nagata's example lies in W but is not tame. It is not known whether W is all of $\mathrm{GA}_3(k)$.

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Department of Mathematics, Washington University in St. Louis, St. Louis, MO 63130 *E-mail*: berson@math.wustl.edu, wright@math.wustl.edu