arXiv:0707.1502v2 [math.GR] 31 Dec 2007

QUASI-ISOMETRIES BETWEEN TUBULAR GROUPS

CHRISTOPHER H CASHEN

Abstract. We give a method of constructing maps between tubular groups inductively according to a finite set of strategies. This map will be a quasi-isometry exactly when the set of strategies satisfies certain consistency criteria. Conversely, if there exists a quasi-isometry between tubular groups, then there is a consistent set of strategies for building a quasi-isometry between them.

For two given tubular groups there are only finitely many candidate sets of strategies to consider, so it is possible in finite time to either produce a consistent set of strategies or decide that such a set does not exist. Consequently, there is an algorithm which in finite time decides whether or not two tubular groups are quasi-isometric.

1. Introduction

One line of attack in Gromov's program to classify finitely generated groups up to quasi-isometry has been to study splittings of groups. Given a splitting of a group G into a graph of groups, one would like to know whether there is a similar splitting for a finitely generated group H quasi-isometric to G, and what constraints such splittings impose upon quasi-isometries between the groups.

A result of this type applies when G and H are accessible groups. Paposoglu and Whyte showed that G and H are quasi-isometric if and only if they have the same number of ends and if, in terminal splittings over finite subgroups, they have the same sets of quasi-isometry classes of one ended vertex groups [10]. In terms of the Bass-Serre trees for such groups, this is a local restriction.

When edge groups are infinite there may be large scale restrictions as well. Consider, for example, graphs of groups where all the local groups are infinite cyclic, such as the Baumslag-Solitar groups. Here there are no obvious local restrictions; all the vertex spaces are quasi-isometric to each other. However, crossing an edge space contributes a *height change*. Quasi-isometries between graphs of $\mathbb{Z}$'s must induce coarsely height preserving quasi-isometries of the corresponding Bass-Serre trees [13].

Perhaps the next most complicated situation to consider would be graphs of groups with infinite cyclic edge groups and vertex group free abelian of rank two. Such a group can be thought of as the fundamental group of a finite 2-complex consisting of a disjoint union of tori glued together by annuli. Martin Bridson has described such groups as "tubular".

Mosher, Sageev, and Whyte prove splitting rigidity results for graphs of coarse Poincaré duality groups under some additional hypotheses [8], [9]. We define a class of tubular groups which satisfy these hypothesis. Such splittings are quasi-isometrically rigid. Furthermore, the patterns of attachment of edge groups to vertex groups are quasi-isometry invariants of the groups. These edge patterns give

Key words and phrases. Tubular Groups.

local quasi-isometry restrictions, and height change gives a large scale restriction, similar to the graph of $\mathbb{Z}$'s example.

We will show that we can cover the Bass-Serre tree of a tubular group by a collection of infinite subtrees, called P–sets, which intersect pairwise in at most a single vertex. We organize the P–sets into a tree of P–sets, which is essentially the nerve of the aforementioned cover. The P–sets are similar to the Bass-Serre trees of Baumslag-Solitar groups, and we can use similar methods construct quasi-isometries between P–set spaces. To construct quasi-isometries of tubular groups we will need to coordinate the quasi-isometries on each P–set space so that they piece together satisfying the local edge-to-vertex pattern rigidity and global height change restrictions.

To accomplish this, we define *sets of strategies* for inductively building quasi-isometries in a way that will satisfy the local restrictions given by edge pattern rigidity. We give consistency criteria for such a set of strategies that determine whether the large scale restrictions on height change are satisfied. Theorem 4.5 shows that there is an algorithm that in finite time will either produce a consistent set of strategies or decide that no such set exists.

We prove a number of results which can be summarized as:

Theorem. *The following are equivalent:*

1. *Tubular groups G_1 and G_2 are quasi-isometric.*
2. *There exists an allowable isomorphism between the trees of P–sets of G_1 and G_2.*
3. *There exists a consistent set of strategies for G_1 and G_2.*

Combining these results with Theorem 4.5, we get he main result of this paper, Theorem 4.9: there is an algorithm that will take graph of groups decompositions of two tubular groups and in finite time decide whether or not the groups are quasi-isometric.

1.1. **Previous Examples of Tubular Groups.** Examples of tubular groups have appeared in the literature in various guises.

The fundamental groups of the Torus Complexes of Croke and Kleiner [7] are examples of tubular groups. These Torus Complexes provided examples of finite, non-positively curved, homeomorphic 2-complexes whose universal covers have non-homeomorphic ideal boundaries.

Right-Angled Artin groups whose defining graphs are trees of diameter at least three are tubular groups. Such groups are also graph-manifold groups and have been shown to be quasi-isometric to each other by Behrstock and Neumann as a special case of their quasi-isometry classification of graph-manifold groups [1]. This result is recovered as Corollary 5.5

Examples of tubular groups were used in work of Brady and Bridson [2], as well as Brady, Bridson, Forester, and Shankar [3], to show that the isoperimetric spectrum is dense in $[2, \infty)$ and that $\mathbb{Q} \cap [2, \infty) \subset IP$, respectively. In the latter work these examples were termed "snowflake groups".

1.2. **Acknowledgements.** This paper is based on a thesis submitted in partial fulfillment of the requirements for the doctoral degree at the Graduate College of the University of Illinois at Chicago, under the supervision of Kevin Whyte. Additional work was conducted during MSRI's Geometric Group Theory Program.

2. Preliminaries

This section contains standard definitions and constructions. Notation will most closely resemble that of Mosher, Sageev, and Whyte [9].

2.1. **Coarse Geometry.** Let (X, d_X) and (Y, d_Y) be metric spaces, and let A, $B \subset X$. The closed r-neighborhood of A in X is denoted $N_r(A)$. The set A is *coarsely contained in* B, $A \overset{c}{\subset} B$, if $\exists r$ such that $A \subset N_r(B)$. Subsets A and B are *coarsely equivalent*, $A \overset{c}{=} B$, if $A \overset{c}{\subset} B$ and $B \overset{c}{\subset} A$. A subset A is *coarsely dense* in X if $A \overset{c}{=} X$.

A subspace C of X is a *coarse intersection* of A and B, $C = A \overset{c}{\cap} B$, if, for sufficiently large r, $C \overset{c}{=} N_r(A) \cap N_r(B)$.

A map $f\colon X \to Y$ is *K-bilipschitz*, for $K \geq 1$, if

$$\frac{1}{K} d_X(x, y) \leq d_Y(f(x), f(y)) \leq K d_X(x, y)$$

for all $x, y \in X$. The map f is a *(K, C)-quasi-isometric embedding*, for $K \geq 1$, $C \geq 0$, if

$$\frac{1}{K} d_X(x, y) - C \leq d_Y(f(x), f(y)) \leq K d_x(x, y) + C$$

for all $x, y \in X$. Furthermore, f is a *(K, C)-quasi-isometry* if it is a (K, C)-quasi-isometric embedding and the image is C-coarsely dense in Y.

Two maps $f, g\colon X \to Y$ are *bounded distance* from each other, $f \overset{c}{=} g$, if there is a $C \geq 0$ such that, $\forall x \in X$, $d_Y(f(x), g(x)) \leq C$. Two maps, $f\colon X \to Y$ and $\bar{f}\colon Y \to X$, are *coarse inverses* if $f \circ \bar{f} \overset{c}{=} Id_Y$ and $\bar{f} \circ f \overset{c}{=} Id_X$. If f is a quasi-isometry, there is a coarse inverse, $\bar{f}\colon Y \to X$, of f which is also a quasi-isometry, with constants depending on those of f.

2.2. **Bass-Serre Theory.** If Γ is a graph, let $\mathcal{V}\Gamma$ denote the vertex set, and $\mathcal{E}\Gamma$ the edge set. Let $\mathcal{VE}\Gamma = \mathcal{V}\Gamma \cup \mathcal{E}\Gamma$. The set of endpoints of Γ is $\mathcal{E}\Gamma \times \{0, 1\}$.

Each edge has two endpoints, of the form $\eta = (e, i)$, where the i should be taken mod 2. Each endpoint is identified with some vertex $v(\eta)$ such that $e = e(\eta)$ is incident to $v(\eta)$.

A *graph of groups*, $(\Gamma, \{G_\gamma\}, \{\phi_\eta\})$, is a graph, Γ, equipped with a *local group* G_γ for each $\gamma \in \mathcal{VE}\Gamma$, and *edge injections* $\phi_\eta \in \mathrm{Hom}(G_{e(\eta)}, G_{v(\eta)})$ for each endpoint η. We will generally use Γ to denote the graph of groups, and refer to the *underlying graph of* Γ if we wish to consider only the graph itself.

Note. A graph of groups is *of finite type* if the underlying graph is finite, the vertex groups are finitely presented, and the edge groups are finitely generated. All the graphs of groups of interest in this paper are of finite type, so, from this point forward, finite type can be taken as an implicit hypothesis for any statement about graphs of groups.

Associated to a graph of groups there is a finitely presented group,

$$G = \pi_1(\Gamma, \{G_\gamma\}, \{\phi_\eta\}),$$

the *fundamental group of the graph of groups* [12].

A graph of groups is *reducible* if there is an edge e such that the vertices $v(e, 0)$ and $v(e, 1)$ are distinct, and such that one of the edge homomorphisms $\phi_{(e,i)}$ is surjective. In this case it is possible to simplify the graph of groups without changing

the fundamental group. Remove e and $v(e,i)$ from Γ, and for any other endpoint with $v(e',j) = v(e,i)$, replace $\phi_{(e',j)}$ with

$$\phi_{(e,i+1)} \circ \phi_{(e,i)}^{-1} \circ \phi_{(e',j)}$$

Scott and Wall gave a topological realization of G [11]. Build a *graph of spaces*, $\mathcal{K}$, for $(\Gamma, \{G_\gamma\}, \{\phi_\eta\})$ by choosing *local spaces*, $\mathcal{K}_\gamma$, for each $\gamma \in \mathcal{VE}\Gamma$. For each γ, choose $\mathcal{K}_\gamma$ to be a pointed, connected, compact CW-complex, with a map $\pi_1(\mathcal{K}_\gamma) \to G_\gamma$. This map should be an isomorphism if $\gamma \in \mathcal{V}\Gamma$, and an epimorphism if $\gamma \in \mathcal{E}\Gamma$. For each endpoint η of Γ, choose an *edge map*, a pointed CW-map $f_\eta \colon \mathcal{K}_{e(\eta)} \to \mathcal{K}_{v(\eta)}$, such that the induced map on fundamental groups agrees with the edge injection.

Now, let $\mathcal{K}$ be the finite CW-complex obtained from the disjoint union

$$\coprod_{v \in \mathcal{V}\Gamma} \mathcal{K}_v \amalg \coprod_{e \in \mathcal{E}\Gamma} \mathcal{K}_e \times [0,1]$$

by using $f_{(e,i)}$ to glue $\mathcal{K}_{e \times \{i\}}$ to $\mathcal{K}_{v(e,i)}$, for each endpoint (e,i). The fundamental group $\pi_1(\mathcal{K})$ is well defined, up to isomorphism, and, by van Kampen's Theorem, is isomorphic to G.

Consider the universal cover $X = \tilde{\mathcal{K}}$, with covering map p and metric lifted from $\mathcal{K}$. The group G acts properly discontinuously and cocompactly by isometries by deck transformations on X, so they are quasi-isometric by the Švarc-Milnor Lemma [5]. Thus, X serves as a *geometric model* for G. For questions of the coarse geometry of G, it is sufficient to study X.

The space X can be decomposed into path connected lifts of the local spaces $\mathcal{K}_\gamma$, and the action of G on X respects this decomposition. Let $D\Gamma = q(X)$ be the quotient space of the decomposition. The quotient is a tree on which G acts without edge inversion, and $D\Gamma$ is G-equivariantly isomorphic to the Bass-Serre Tree of Γ, which is also known as the development of Γ. Call X the *Bass-Serre Complex*, and $X \xrightarrow{q} D\Gamma$ the *Bass-Serre tree of spaces* for Γ.

For $v \in \mathcal{V}D\Gamma$, $X_v = q^{-1}(v)$ is called a *vertex space*, and $\mathcal{V}X = \bigcup_{v \in \mathcal{V}D\Gamma} X_v$ is the set of vertex spaces. The set of vertex spaces is $\frac{1}{2}$-coarsely dense in X. For $e \in \mathcal{E}D\Gamma$, $X_e = q^{-1}(\text{midpoint of } e)$ is called an *edge space*.

For $t \in D\Gamma$, $\mathrm{Stab}_G(t) = \mathrm{Stab}_G(X_t)$. The group $\mathrm{Stab}_G(t)$ is conjugate in G to $G_{p(t)}$, and $\mathrm{Stab}_G(t)$ acts on X_t as the deck transformation group of the covering map $X_t \to \mathcal{K}_{p(t)}$.

3. Tubular Groups

3.1. **Definitions and Rigidity Results.** The motivating examples of tubular groups are graphs of groups with edge groups $\mathbb{Z}$ and vertex groups $\mathbb{Z}^2$. However, this description is not sufficient to give a quasi-isometrically closed class of groups.

Definition 3.1. A *tubular group* is the fundamental group of a finite, connected graph of groups satisfying the following conditions:

(1) Every edge group is finitely generated and quasi-isometric to $\mathbb{Z}$.
(2) Every vertex group is finitely generated and quasi-isometric to either $\mathbb{Z}$ or $\mathbb{Z}^2$.
(3) There is at least one vertex group quasi-isometric to $\mathbb{Z}^2$.
(4) For every vertex quasi-isometric to $\mathbb{Z}^2$, the "crossing graph condition" is satisfied.

Remarks.

(1) Graphs of groups with all local groups quasi-isometric to $\mathbb{Z}$ have been classified by Whyte [13].
(2) Condition (4) is satisfied if every vertex of the Bass-Serre tree whose stabilizer is quasi-isometric to $\mathbb{Z}^2$ has incident edges whose edge spaces are not coarsely contained in one another. This will be discussed further in Section 3.1.1.
(3) Suppose $v \in \mathcal{V}\Gamma$ with G_v quasi-isometric to $\mathbb{Z}$. Suppose $e \in \mathcal{E}\Gamma$ such that $v(e,i) = v$ and $v(e,i+1) \neq v$. We can assume that $\phi_{(e,i)}(G_e)$ is a subgroup of G_v of index at least two. Otherwise the graph of groups is reducible and could be simplified without changing the fundamental group.

3.1.1. *Rigidity Results of Mosher-Sageev-Whyte.* In this subsection we recall results of Mosher, Sageev, and Whyte [9] and apply them to tubular groups.

In a graph of groups, a *depth zero* vertex group is one which is not strictly coarsely contained in any other vertex group. In a tubular group, those are precisely the vertex groups quasi-isometric to $\mathbb{Z}^2$.

Let $\mathcal{V}_0 D\Gamma$ be the set of depth zero vertices of $D\Gamma$. Let $\mathcal{V}_0 X = \bigcup_{v \in \mathcal{V}_0 D\Gamma} X_v$.

Given a graph of groups, G, with Bass-Serre tree of spaces $X \to T$, the following hypothesis will be needed:

(1) G is finite type, irreducible, and finite depth.
(2) No depth zero raft of the Bass-Serre tree T is a line.
(3) Each depth zero vertex group is coarse PD.
(4) The crossing graph condition holds for each depth zero vertex of T which is a raft.
(5) Each vertex and edge group of G is coarse finite type.

Proposition 3.2. *Tubular groups satisfy these hypothesis.*

Proof. The depth zero vertices are all rafts, and these are the only depth zero rafts. Virtually abelian groups are coarse PD and coarse finite type. □

Theorem 3.3 (Quasi-isometric Rigidity Theorem)**.** [9, Theorem 1.5] *Let G be a graph of groups satisfying (1)-(5) above. If H is a finitely generated group quasi-isometric to $\pi_1 G$ then H is the fundamental group of a graph of groups satisfying (1)-(5).*

Theorem 3.4 (Quasi-isometric Classification Theorem)**.** [9, Theorem 1.6] *Let G, G' be graphs of groups satisfying (1)-(5) above. Let $X \to T$, $X' \to T'$ be Bass-Serre tree of spaces for G, G', respectively. If $f\colon X \to X'$ is a quasi-isometry then f coarsely respects vertex and edge spaces. To be precise, for any $K \geq 1$, $C \geq 0$ there exists a K', C' quasi-isometry $f_\#\colon \mathcal{VE}(T) \to \mathcal{VE}(T')$ such that the following hold:*

- *If $a \in \mathcal{VE}(T)$ then $d_{\mathcal{H}}(f(X_a), X'_{f_\#(a)}) \leq C'$*
- *If $a' \in \mathcal{VE}(T')$ then there exists $a \in \mathcal{VE}(T)$ such that $d_{\mathcal{H}}(f(X_a), X'_{a'}) \leq C'$*

Corollary 3.5. *The class of tubular groups is closed under quasi-isometry. That is, any finitely generated group quasi-isometric to a tubular group is itself a tubular group. Furthermore, any quasi-isometry between tubular groups coarsely respects vertex and edge spaces.*

Note that in a tubular group, a vertex space quasi-isometric to $\mathbb{Z}^2$ is not bounded Hausdorff distance from any other vertex space. Thus, we can change a quasi-isometry by a bounded amount so that it actually respects such vertex spaces.

If $V \subset \mathbb{R}^n$ is a linear subspace, let P_V be the set of affine subspaces of $\mathbb{R}^n$ parallel to V. For a finite collection, F, of linear subspaces, the *affine pattern* induced by F, P_F, is the union of the P_V for $V \in F$. An affine pattern is *rigid* if for every K, C, R there is an R' such that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a K, C quasi-isometry which R-coarsely respects each P_V, then f is within R' of an affine homothety.

Lemma 3.6. [9, Lemma 7.2] *If F is a finite collection of linear subspaces of $\mathbb{R}^n$ which contains $n+1$ hyperplanes in general position, then F is rigid.*

In particular, a collection of at least three distinct lines in the plane is rigid.

Corollary 3.7. [9, Corollary 7.11] *Let Γ be a graph of groups with all vertex and edge groups finitely generated abelian groups. Assume that for each depth zero, one vertex raft v in the Bass-Serre tree, the collection of edge spaces at the vertex space of v is a rigid affine pattern. Assume also that there are no line-like rafts of depth zero. If H is any finitely generated group quasi-isometric to $G = \pi_1(\Gamma)$, then H splits as a graph of virtually abelian groups and the quasi-isometry $G \to H$ is affine along each depth zero, one vertex raft. Moreover, the set of affine equivalence classes of edge patterns is the same for H as it is for G.*

3.2. **Geometric Models for Tubular Groups.**

3.2.1. *Affine Patterns of Lines in the Plane.* Let $F = \{l_1, l_2, .., l_n\}$, $n \geq 3$, be a finite collection of distinct lines through the origin in $\mathbb{R}^2$, with the usual Euclidean metric. Two such collections, F and F', are *linearly equivalent* if there exists $A \in \mathrm{GL}_2\mathbb{R}$ such that $AF = \{Al_i\} = F'$. Scalar matrices in $\mathrm{GL}_2\mathbb{R}$ will preserve any such F, so we projectivise.

Let $PF = \{m_1, m_2, .., m_n\}$ be a collection of slopes in $\mathbb{RP}$. There is a finite subgroup, $L_{PF} \subset \mathrm{PGL}_2\mathbb{R}$, which fixes PF set-wise.

Lift L_{PF} to $\mathrm{GL}_2(\mathbb{R})$ by taking $L_F = L_{PF}\,\mathrm{Z}(\mathrm{GL}_2\mathbb{R}) \cap \{A \in \mathrm{GL}_2\mathbb{R} \mid \det(A) = \pm 1\}$. We call L_{PF} the *group of symmetries* of F, and call F *symmetric* if L_F acts by isometries.

Proposition 3.8. *For any collection F of $n \geq 3$ distinct lines through the origin, there is a symmetric representative of the linear equivalence class of F. Choosing an isometry class of symmetric representative is equivalent to choosing a Euclidean metric on $\mathbb{R}^2$ for which F is symmetric.*

Proof. L_F is finite. Define a new metric on the plane by

$$\langle x, y\rangle_{L_F} = \frac{1}{|L_F|}\sum_{A \in L_F} \langle Ax, Ay\rangle\,.$$

The group L_F acts isometrically on the plane with this metric. There is an $A_F \in \mathrm{GL}_2\mathbb{R}$ such that $\langle x, y\rangle_{L_F} = \langle A_F x, A_F y\rangle$, so $A_F F$ is a symmetric representative of the linear equivalence class of F.

Conversely, suppose F' is a symmetric representative for the linear equivalence class of F. Suppose A_1 and A_2 are matrices such that $|\det(A_i)| = \pm 1$ and $A_i F = F'$. Then $A_1 A_2^{-1} \in L_{F'}$. Since F' is symmetric, this means the A_i differ by an isometry, so

$$\langle A_1 x, A_1 y\rangle = \langle A_2 x, A_2 y\rangle\,. \qquad \square$$

For $n = 2$ we will consider two orthogonal lines to be a symmetric pattern, but the group of symmetries in this case is not a finite group.

For $n = 3$ there is a single linear equivalence class. There is, up to isometry, a unique symmetric representative, consisting of three lines meeting at angles $\frac{\pi}{3}$. The group of symmetries is isomorphic to S_3.

For $n = 4$ there are infinitely many linear equivalence classes, indexed by the cross ratios $[-1, 0)$. Each has, up to isometry, a unique symmetric representative, and a transitive group of symmetries. When the cross ratio is -1 the group of symmetries is isomorphic to the dihedral group of order 8. Otherwise, the group of symmetries is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

For $n = 5$ there is no longer a unique symmetric representative for every linear equivalence class. Indeed, there are five-line patterns with trivial group of symmetries. In such a class, every member is symmetric, so the pattern does not determine a canonical choice of metric on the plane.

3.2.2. *Coarse Bass-Serre Complex.* It is possible to relax some of the group theoretic restrictions from Section 2.2 and still get a geometric model quasi-isometric to $G = \pi_1\Gamma$. Following the proof of Lemma 2.9 of [9], we construct a *coarse Bass-Serre complex* $X' \to D\Gamma$ to serve as a geometric model for G.

Let Γ be a graph of groups for a tubular group $G = \pi_1\Gamma$.

Let v be a depth zero vertex of Γ, and e an edge of Γ incident to v at endpoint η.

Suppose A is a group quasi-isometric to $\mathbb{Z}^n$. The group A has a finite index normal subgroup $B \cong \mathbb{Z}^n$ [4]. There is a quasi-isometry, $f_B \colon A \to B$, at bounded distance from Id_A, with $f_B|_B = Id_B$. Furthermore, f_B takes cyclic subgroups of A to within bounded distance of cyclic subgroups of B.

Applying this reasoning to G_v and G_e,

$$\begin{array}{ccc} G_e & \xrightarrow[qi]{f_{\mathbb{Z}}} & \mathbb{Z} \\ {\scriptstyle \phi_\eta}\downarrow & & \downarrow \\ G_v & \xrightarrow[qi]{f_{\mathbb{Z}^2}} & \mathbb{Z}^2 \end{array}$$

the image of the map $f_{\mathbb{Z}^2} \circ \phi_\eta \circ \bar{f}_{\mathbb{Z}}$ is bounded distance from a cyclic subgroup

$$\langle x^a y^b \rangle < \langle x, y \rangle \cong \mathbb{Z}^2.$$

The usual inclusion of $\mathbb{Z}^2$ into $\mathbb{R}^2$ is a quasi-isometry, and the subgroup $\langle x^a y^b \rangle$ includes into the line through the origin with rational slope $\frac{b}{a}$. In this way, each edge incident to v is associated to a line in $\mathbb{R}^2$. Let F_v be the set of distinct lines. The affine pattern induced by this set is called the *edge pattern*, and is well defined up to linear equivalence. If F_v contains n distinct lines, we will say that v *has* n *lines* or *is an* n*-line vertex*. Furthermore, we can choose a new Euclidean metric on $\mathbb{R}^2$ to make this pattern symmetric.

Let $X \xrightarrow{q} D\Gamma$ be a Bass-Serre tree of spaces for Γ. For each $\gamma \in \mathcal{V}\mathcal{E}D\Gamma$, let $h_\gamma \colon X_\gamma \to X'_\gamma$, where X'_γ is either $\mathbb{R}$ or $\mathbb{R}^2$ and h_γ is the quasi-isometry given by restricting to a finite index normal abelian subgroup and then including into X'_γ.

For each vertex $v \in \mathcal{V}D\Gamma$, and each incident edge $e \in \mathcal{E}D\Gamma$, let $F_{ev} \colon X_e \to X_v$ be the attaching map. Define a new attaching map by $F'_{ev} = h_v \circ F_{ev} \circ \bar{h}_e \colon X'_e \to X'_v$.

Define X' by taking the disjoint union

$$\coprod_{v\in\mathcal{V}D\Gamma} X'_v \amalg \coprod_{e\in\mathcal{E}D\Gamma} X'_e \times [0,1]$$

and gluing according to the attaching maps.

The h_γ have uniform quasi-isometry constants, since the underlying graph was finite, so they piece together to give a quasi-isometry $h\colon X \to X'$. Since G was quasi-isometric to X, we now have that G is quasi-isometric to X'. The action of G on X is quasi-conjugated by h to give a proper, cobounded quasi-action of G on X'. The space X' is still a tree of spaces over $D\Gamma$, and $X' \to D\Gamma$ is called a coarse Bass-Serre complex.

When v is a depth zero vertex, the edge spaces of the incident edges attach to X'_v along lines of the edge pattern in X'_v.

3.2.3. *Contraction Factors and Height Change.* Let e be an edge of $D\Gamma$, and let v_i be the vertex at endpoint (e,i) of e, for $i = 0, 1$. For each i, F'_{ev_i} maps $X'_e = \mathbb{R}$ within bounded distance of a line in X'_{v_i}, and there is a factor l_i such that $d_{X'_{v_i}}(F'_{ev_i}(x), F'_{ev_i}(y)) = l_i d_{X'_e}(x,y)$ to within bounded additive error. Define the *contraction factor across* e to be $\frac{l_1}{l_0}$ and the *height change across* e to be

$$h(e) = -\ln(\frac{l_1}{l_0}).$$

Metrize the strip $X'_e \times [0,1]$ as the strip $0 \le y \le 1$ in the plane with metric $(\frac{l_1}{l_0})^{2y}dx^2 + dy^2$, so the edge strips are horostrips in a plane of constant curvature $-(\ln \frac{l_1}{l_0})^2$. Without changing the quasi-isometry type, we can change X' by a bounded amount so that F'_{ev_i} glues $X'_e \times \{i\}$ isometrically along a line in X'_{v_i}.

Let p be vertical projection from the bottom ($i = 0$) of the strip to the top ($i = 1$) of the strip. The map p is closest point projection. If a and b are points in the bottom edge of the strip, and d_i is the distance in $X'_e \times \{i\}$, then

$$\frac{d_1(p(a),p(b))}{d_0(a,b)} = \frac{l_2}{l_1} = \exp(-h(e)).$$

For vertices $v, w \in D\Gamma$, the *height change from* v *to* w, $h(v,w)$, is the sum of the height changes across the edges of the geodesic between v and w. This quantity will sometimes be called the *height of* w *relative to* v.

3.2.4. *Geometric Models for Two Tubular Groups.* The coarse Bass-Serre complex, X', will serve as the geometric model for its tubular group. We will have no further use for the original Bass-Serre complex.

From this point forward, given two tubular groups, $G_i = \pi_1(\Gamma_i)$, for $i = 1, 2$, X will always refer to the geometric model for G_1, and Y will always refer to the geometric model for G_2.

There were choices involved in metrizing the geometric models. For each depth zero vertex in the underlying graph we chose a symmetric representative for the equivalence class of affine pattern induced by the edge inclusions. It will be convenient to assume that these choices were consistent. That is, if vertex spaces of X and Y have equivalent affine patterns of edge inclusions, assume that we have chosen the same symmetric representative for each of them.

3.2.5. *Examples of Tubular Groups.* There is only one affine equivalence class of 3-line pattern. Tubular groups whose depth zero vertices all have three line patterns are of particular interest since edge pattern rigidity will provide no quasi-isometry obstructions.

Let F be the set of lines through the origin of distinct slopes 0, $\frac{s}{r}$, and $\frac{u}{t}$. Let

$$A' = \begin{pmatrix} 1 & -\frac{1}{2}\frac{ru+st}{su} \\ 0 & \frac{\sqrt{3}}{2}\frac{ru-st}{su} \end{pmatrix}.$$

Let $A = \frac{1}{\sqrt{|\det(A')|}}A'$. Pull back the metric on $\mathbb{R}^2$ according to the matrix A, as in Proposition 3.8. This makes the three-line pattern generated by F symmetric.

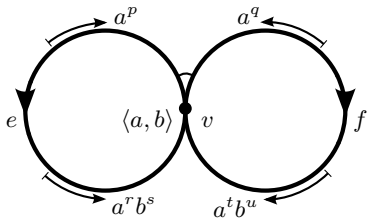

FIGURE 1. A one torus group.

Let Γ be the graph of groups in Figure 1.

$$G = \pi_1\Gamma = \left\langle a, b, x, y \mid [a,b] = 1,\, xa^px^{-1} = a^rb^s,\, ya^qy^{-1} = a^tb^u \right\rangle$$

Let $\nu = \left|\frac{ru-st}{su}\right|$. The contraction factor across e is

$$\lambda = \frac{|A\binom{r}{s}|}{|A\binom{p}{0}|} = \left|\frac{s}{p}\right|\nu.$$

The contraction factor across f is

$$\mu = \frac{|A\binom{t}{u}|}{|A\binom{q}{0}|} = \left|\frac{u}{q}\right|\nu.$$

A special case is a group $G_{p,r}$ where $p = q > r = t > 0$, $s = 1$, and $u = -1$. For such a group the contraction factors across the two edges are

$$\lambda = \mu = \frac{2r}{p},$$

and $0 < \lambda < 2$.

Brady and Bridson have shown [2] that the group $G_{p,r}$ has Dehn function

$$n^{4-2\log_2\lambda}.$$

This proved that there are no gaps in the isoperimetric spectrum beyond 2.

Proposition 3.9. *Suppose we are given positive rational contraction factors λ and μ and four indices of edge inclusions, p, q, $m = \gcd(r,s)$, and $n = \gcd(t,u)$. These can be realized in a tubular group as in Figure 1 if and only if, as reduced rationals, $\frac{\mu q}{n} = \frac{i}{j}$ and $\frac{\lambda p}{m} = \frac{i}{k}$ where i, j, and k are pairwise coprime.*

Proof. If these constants did arise from a tubular group, the matrix A' which was used to define the metric on the torus allows us to realize the group G_v as a lattice in $\mathbb{R}^2$. In this lattice, $(1,0)$, $\frac{\lambda p}{m}(1/2, \sqrt{3}/2)$, and $\frac{\mu q}{n}(-1/2, \sqrt{3}/2)$ must be primitive elements, since these are the images of generators of maximal cyclic subgroups in G. These three elements generate a lattice in which each of them is primitive only if, as reduced rationals, $\frac{\lambda p}{m}$ and $\frac{\mu q}{n}$ have the same numerators and relatively prime denominators.

For the converse, choose any $g \in \mathbb{Z}$ such that

$$g \equiv -\frac{1}{j} \bmod k.$$

Choose any h coprime to g, $\frac{1+gj}{k}$, and i.

Let $r = mi\left(\frac{1+gj}{k}\right)$, $s = mhj$, $t = ngi$, and $u = nhk$.

With these choices, the graph of groups in Figure 1 gives the desired group. □

Corollary 3.10. *It is possible to construct a tubular group as in Figure 1 with an arbitrary pair of positive rationals as the contraction factors.*

3.3. **P–sets.** No two depth zero vertex spaces are bounded Hausdorff distance from one another. However, there is a weaker characterization of closeness of two local spaces according to whether they are close on unbounded subsets. We define the corresponding relation on the vertices and edges of the Bass-Serre tree:

Definition 3.11. Two elements $x, y \in \mathcal{VE}D\Gamma$ *satisfy Relation P* if $X_x \overset{c}{\cap} X_y$ is unbounded.

Quasi-isometries respect vertex and edge spaces, and preserve boundedness and unboundedness, so Relation P is invariant under quasi-isometry.

Definition 3.12. A *P–set* of $D\Gamma$ is a maximal subset of $\mathcal{VE}D\Gamma$ such that any two elements satisfy Relation P.

Proposition 3.13 (Properties of P–sets)**.**

1. *A P–set is a subtree of $D\Gamma$.*
2. *Every depth zero vertex of a P–set is adjacent to infinitely many other vertices in that P–set.*
3. *An n-line, depth zero vertex belongs to n distinct P–sets.*
4. *Edges and positive depth vertices belong to exactly one P–set.*
5. *Any two P–sets are either disjoint or intersect in exactly one vertex, which is necessarily a depth zero vertex.*

Proof. Let e_1 and e_2 be edges in $D\Gamma$ incident to a common depth zero vertex, v.

The vertex v satisfies Relation P with either of the e_i.

The edges e_1 and e_2 satisfy Relation P if and only if their edge strips glue to X_v along parallel lines.

The list of properties follows easily from this observation □

The fundamental group $G = \pi_1(\Gamma)$ quasi-acts on X. Quasi-isometries preserve Relation P, so the action of G on $D\Gamma$ induces an action of G on the set of P–sets of $D\Gamma$.

Definition 3.14. Within a P–set S, two depth zero vertices v and w are *of the same type* if $\exists g \in \mathrm{Stab}_G(S)$ such that $gv = w$.

Type is an equivalence relation among depth zero vertices of a fixed P–set. For an n-line depth zero vertex, the equivalence class of the vertex under the action of the whole group splits into at most n vertex types in S.

If there is a height change between vertices of the same type in a P–set, the geodesic segment joining them projects to a non-trivial loop in the underlying graph of Γ. The group element corresponding to this loop is an infinite order element of the P–set stabilizer. Iterating the action of this group element gives vertices of the P–set all of the same type occurring at an unbounded set of heights.

If there is zero height change between every pair of vertices of the same type in a P–set then, since there are only finitely many vertex types, the vertices of the P–set occur at only finitely many heights. Thus, there is bounded height change between any two vertices of the P–set.

For each orbit of P–sets, pick a representative S_i and fix an ordered list of representative $x_{i,j}$ for the vertex types in S_i. Suppose S and R are P–sets, x and y are vertices in R, $g \in G$ such that $gR = S$, and $k \in \mathrm{Stab}_G(R)$ such that $kx = y$. Then $gkg^{-1} \in \mathrm{Stab}_G S$ and $gkg^{-1}gx = gkx = gy$, so the action of the group takes vertices of the same type in one P–set to vertices of the same type in another. Thus, we can fix an ordering of vertex types for each equivalence class of P–set.

We will say that $x \in \mathcal{V}_0 S$ is of type $\{[S_i], j\}$ with respect to S if there is some $g \in G$ such that $gS = S_i$ and $gx = x_{i,j}$.

Proposition 3.15. *Suppose $v \in \mathcal{V}_0 D\Gamma$, and R and S are P–sets containing v. There is an element $g \in \mathrm{Stab}_G(v)$ such that $gR = S$ if and only if v is of the same type with respect to both R and S.*

Proof. Suppose $\exists g \in \mathrm{Stab}_G(v)$ such that $gR = S$. The P–sets R and S then belong to the same equivalence class; call it $[S_i]$. Suppose $v \in \{[S_i], j\}$ with respect to R and $v \in \{[S_i], k\}$ with respect to S. Then there are elements $f, h \in G$ with $fR = S_i$, $fv = x_{i,j}$, $hS = S_i$, and $hv = x_{i,k}$. However, this would mean that $fgh^{-1} \in \mathrm{Stab}_G(S_i)$ and $fgh^{-1}x_{i,k} = x_{i,j}$, but this is only possible if $j = k$.

Conversely, suppose $v \in \{[S_i], j\}$ with respect to both R and S. Then there are elements $f, h \in G$ with $fR = S_i$, $fv = x_{i,j}$, $hS = S_i$, and $hv = x_{i,j}$, so $g = h^{-1}f$ fixes v and takes R to S.

□

Definition 3.16. The *tree of P–sets,* T_Γ, of a tubular group, $G = \pi_1(\Gamma)$, is given by:

- $\mathcal{V}T_\Gamma = \mathcal{V}_0 D\Gamma \amalg \{\text{P–sets of } D\Gamma\}$
- Edges are determined by inclusion of vertices in P–sets. Each edge is assigned length $\frac{1}{2}$.

The collection of P–sets is a cover of $D\Gamma$ by subtrees which intersect pairwise in at most a single vertex. The nerve of such a cover is simply connected, and T_Γ is a deformation retraction of this nerve. So, T_Γ is, in fact, a tree.

It will be convenient to extend the notion of height change across an edge to some of the edges of T_Γ. If R is a P–set with bounded height change, pick some adjacent depth zero vertex, v, of maximal height. For any vertex w adjacent to R, define $h(w, R) = h(w, v)$.

The action of G on $D\Gamma$ induces an action of G on T_Γ.

4. Quasi-isometries Among Tubular Groups

4.1. **Allowable Isomorphisms of Trees of P-Sets.** Suppose $G_1 = \pi_1(\Gamma_1)$ and $G_2 = \pi_1(\Gamma_2)$ are tubular groups, with trees of P–sets T_1 and T_2, respectively.

A quasi-isometry, $\psi\colon G_1 \to G_2$ induces a tree isomorphism $\psi_\#\colon T_1 \to T_2$.

Recall that the P–sets $R_1, \dots, R_n$ adjacent to a depth zero vertex v are in one-to-one correspondence with the families of parallel lines forming the edge pattern in the vertex space of v. To satisfy edge pattern rigidity, the bijection

$$\{R_1, \dots, R_n\} \to \{\psi_\#(R_1), \dots, \psi_\#(R_n)\}$$

must correspond to an affine equivalence between the edge patterns of X_v and $Y_{\psi_\#(v)}$.

If this restriction is satisfied for every depth zero vertex, then $\psi_\#$ will be called *locally allowable*.

Define a *rigid component* of T_i to be a connected component of

$$T_i \setminus \{\text{2-line, depth zero vertices in } T_i\}$$

An induced isomorphism $\psi_\#$ must take 2-line, depth zero vertices to 2-line, depth zero vertices, so $\psi_\#$ takes rigid components to rigid components.

Lemma 4.1. *A quasi-isometry between tubular groups induces an isomorphism between their trees of P–sets which is coarsely height preserving on rigid components.*

Proof. Within a P–set space this is exactly the same as Whyte's proof for the graph of $\mathbb{Z}$'s case [13, Lemma 4.1]. The metrics have been chosen so that the restriction of a quasi-isometry to a vertex space with at least three lines expands by the same factor in all directions. This allows us to extend Whyte's argument to the entire rigid component. □

Definition 4.2. A tree isomorphism $\phi\colon T_1 \to T_2$ is *allowable* if it is locally allowable and coarsely height preserving on rigid components.

Lemma 4.1 and the preceding material then give the following corollary.

Corollary 4.3. *Quasi-isometries of tubular groups induce allowable isomorphisms on their trees of P–sets.*

4.2. **Finding Allowable Isomorphisms.** Let $[R_1], \dots, [R_m]$ be the equivalence classes of P–sets in $D\Gamma_1$ under the action by G_1.

Let $[S_1], \dots, [S_n]$ be the equivalence classes of P–sets in $D\Gamma_2$ under the action by G_2.

Let $\rho(a)$ be the number of vertex types in $[R_a]$ and let $\sigma(b)$ be the number of vertex types in $[S_b]$.

A *match* is a pair $([R], [S])$ consisting of an equivalence class of P–set from $D\Gamma_1$ and one from $D\Gamma_2$.

An *extension* for $([R_a], [S_b])$ is a $\rho(a) \times \sigma(b)$ matrix, (e_{ij}), with entries described below. An extension must have at least one non-zero entry in each row and in each column.

Recall that we had chosen representatives $x_{a,i} \in \{[R_a], i\}$ with respect to R_a and $y_{b,j} \in \{[S_b], j\}$ with respect to S_b.

Consider the set $\{\{[R^l], k_l\}\}_{l=1}^p$, where $x_{a,i}$ is of type $\{[R^l], k_l\}$ with respect to R^l. Similarly, consider the set $\{\{[S^l], k'_l\}\}_{l=1}^p$, where $y_{b,j}$ is of type $\{[S^l], k'_l\}$ with respect to S^l. A linear equivalence between the edge patterns in $X_{x_{a,i}}$ and $Y_{y_{b,j}}$

gives a bijection between P–sets adjacent to $x_{a,i}$ and $y_{b,j}$, which in turn gives a bijection of these sets of types. We will let e_{ij} be any bijection of these sets which can be induced by a linear equivalence of edge patterns and which includes $\{[R_a], i\} \to \{[S_b], j\}$.

Remark. In a torsion free tubular group the set of types a depth zero vertex takes has no duplicate entries. Specifying a type picks out a parallel family of lines in the vertex space. This may fail for groups with torsion, but Proposition 3.15 shows that it is just the ordering of the sets that is ambiguous. In either case, there are at most two distinct bijections to consider.

If there are no such bijections then $e_{ij} = 0$.

Notice that the number of possible extensions for $([R_a], [S_b])$ is bounded above by

$$3^{\rho(a)\sigma(b)}.$$

An extension provides instructions for extending a tree isomorphism in a locally allowable way. Suppose $R \in T_1$ and $S \in T_2$ are P–sets and we have constructed a map sending R to S. To extend the map, choose any bijection $\mathcal{V}_0 R \to \mathcal{V}_0 S$ such that vertices of type $\{[R], i\}$ map to vertices of type $\{[S], j\}$ if and only if $e_{ij} \neq 0$. The value of e_{ij} then tell how to extend to the next level of P–sets. For the purposes of tree isomorphisms, the particular bijection $\mathcal{V}_0 R \to \mathcal{V}_0 S$ does not matter, but more care will be required when building quasi-isometries in Section 4.3.

Conversely, given an allowable isomorphism $\phi\colon N_1(R) \to N_1(S)$, it is possible to "read off" an extension for $([R], [S])$. Set $e_{ij} = 0$ if $\phi(\{[R], i\}) \cap \{[S], j\} = \emptyset$. For every i and j such that $\phi(\{[R], i\}) \cap \{[S], j\} \neq \emptyset$, choose some representative vertex $v_{i,j}$ with $v_{i,j} \in \{[R], i\}$ and $\phi(v_{i,j}) \in \{[S], j\}$. Set e_{ij} equal to the bijection of vertex types of $v_{i,j}$ to vertex types of $\phi(v_{i,j})$ induced by ϕ. Then the matrix (e_{ij}) is an extension for $([R], [S])$.

4.2.1. *Strategies.* A *strategy,* $\mathcal{S}$, for $([R], [S])$ consists of:

(1) a root vertex, $\mathrm{ROOT}(\mathcal{S})$, which is labeled by $([R], [S])$,
(2) an extension $\mathcal{E} = (e_{ij})$ for $([R], [S])$,
(3) a collection of terminal vertices corresponding to the set $\{(\mathcal{M}, E)\}$ of induced matches with height errors coming from $\mathcal{E}$.

The *label* of a terminal vertex is the match associated to it. A strategy can be simplified by considering at most three terminal vertices for each label: one with maximum height error, one with minimum height error, and one with undefined height error. Thus, we can assume that the number of terminal vertices of any strategy is at most $3mn$.

A strategy records how the boundary of a neighborhood of a P–set R in T_1 maps to the boundary of a neighborhood of S in T_2 when mapped according to an extension $\mathcal{E}$ for $([R], [S])$.

If $\mathcal{S}$ is a strategy, define

$$\mathrm{TERM}(\mathcal{S}) = \{\text{terminal vertices of } \mathcal{S}\}$$

A *set of strategies* will consist of a graph, each of whose vertices is labeled by a match and two (not necessarily distinct) strategies for the match, one called the positive strategy and the other called the negative strategy. There will be at most one vertex labeled by a given match, so at most mn vertices.

For each strategy in the set, add an edge to the graph from the root vertex of the strategy to each terminal vertex of the strategy. Label the edge by the sign of the strategy and the height error, if defined, for the appropriate terminal vertex.

4.2.2. *Consistency.* We need to check that building according to the set of strategies will not create unbounded height error. It is necessary that each match consists of two P–sets of bounded height change or two P–sets of unbounded height change, but we also must control accumulation of height error created by the strategies.

For P–sets of bounded height error, the type of a vertex determines its relative height. Choosing an extension therefore determines height errors in a neighborhood of the P–set.

We do not need to worry about height change when it comes to P–sets of unbounded height change; they have enough flexibility to correct any height error which may occur.

Let $\mathcal{M}_1, \dots$ be a list of the matches of P–sets of bounded height change which label vertices of $\mathcal{G}$. For each i, add to the set of inequalities:

$$L_i \leq E_i \leq U_i$$

Let $\mathcal{S}_i^+$ and $\mathcal{S}_i^-$ be the positive and negative strategies chosen for $\mathcal{M}_i$.

Suppose $v \in \mathrm{TERM}(\mathcal{S}_i^+)$ has a defined height error, and the label of v is $\mathcal{M}_j$. Let E be the height error of v in $\mathcal{S}_i^+$. Add the following inequalities to the set:

$$M_i + E \geq L_j$$
$$U_i + E \leq U_j$$

Suppose $v \in \mathrm{TERM}(\mathcal{S}_i^-)$ has a defined height error, and the label of v is $\mathcal{M}_j$. Let E be the height error of v in $\mathcal{S}_I^-$. Add the following inequalities to the set:

$$M_i + E \leq U_j$$
$$L_i + E \geq L_j$$

If the system of inequalities has a solution then height error is well controlled. The L_i and U_i provide bounds.

If such a system of inequalities has a solution, then it has a solution such that all the U_i are positive, all the L_i are negative, and all the M_i are zero. Furthermore, given any fixed B, there is a solution such that, for all i, $U_i > B$ and $L_i < -B$.

Note that if a positive strategy for a match $\mathcal{M}_i$ adds an edge with negative height error that leads back to $\mathcal{M}_i$, then the set of strategies can not be consistent. Similarly, a negative strategy should not create a length one loop with a positive height error.

Lemma 4.4 (Deriving a Set of Strategies)**.** *If there is an allowable isomorphism between T_1 and T_2, then a consistent set of strategies exists.*

Proof. Suppose ϕ is an allowable isomorphism $T_1 \to T_2$ of trees of P–sets.

For each match $([R], [\phi(R)])$ with $[R]$ and $[\phi(R)]$ of unbounded height change, pick some representative R and read off a strategy. This strategy suffices for both the positive and negative strategy of $([R], [\phi(R)])$.

If there is a P–set R of bounded height change such that every depth zero vertex of R has two lines then read off a strategy for R. This strategy suffices for both the positive and negative strategy of $([R], [\phi(R)])$.

The only matches left are pairs of P–sets of bounded height change which contain at least one depth zero vertex with at least 3 lines. Pick such a P–set, R, and depth zero vertex, v_0.

Every rigid component, $\mathcal{C}$, of T_1 contains a vertex, $v_{\mathcal{C}}$, closest to v_0. If $v_{\mathcal{C}} \neq v_0$ then $v_{\mathcal{C}}$ is necessarily a 2-line, depth zero vertex.

For any depth zero vertex, w, in the rigid component $\mathcal{C}$, define $h(w) = h(v_{\mathcal{C}}, w)$.

Make the corresponding definitions for heights of depth zero vertices of T_2.

Let $\mathcal{M}_1, \ldots, \mathcal{M}_a$ be a list of matches which occur as $([R], [\phi(R)])$, with R and $\phi(R)$ of bounded height change and having a depth zero vertex with at least 3 lines. Let

$$U_i = \sup\{h(R) - h(\phi(R)) \mid ([R], [\phi(R)]) = \mathcal{M}_i\}$$
$$L_i = \inf\{h(R) - h(\phi(R)) \mid ([R], [\phi(R)]) = \mathcal{M}_i\}$$

These quantities exist since ϕ is uniformly coarsely height preserving on rigid components.

For any match there are only finitely many possible strategies. For each $1 \leq i \leq a$, if U_i is achieved, pick an R with $U_i = h(R) - h(\phi(R))$ and $([R], [\phi(R)]) = \mathcal{M}_i$ and read off a strategy $\mathcal{S}_i^-$ for $([R], [\phi(R)])$. If U_i is not achieved, let $\mathcal{S}_i^-$ be any strategy which occurs for R with height error arbitrarily close to U_i. These will be the negative strategies.

Pick positive strategies $\mathcal{S}_i^+$ analogously using the L_i.

The set of strategies chosen in this way is consistent, although the height error bounds may be larger than for ϕ. □

Theorem 4.5. *There is an algorithm which in finite time either produces a consistent set of strategies for two trees of P–sets or decides that no such set exists. Furthermore, the algorithm is guaranteed to succeed if the trees of P–sets came from quasi-isometric tubular groups.*

Proof. A set of strategies is completely determined by the matches in it and the choices of extensions for those matches. If T_1 and T_2 are as above, then there are at most mn matches and each extension has at most $\max\{\rho(a)\} \times \max\{\sigma(b)\}$ entries. Each entry takes one of at most 3 values, so we have at most

$$3^{2mn \times \max\{\rho(a)\} \times \max\{\sigma(b)\}}$$

possible candidate sets of strategies.

Enumerate this list of candidates and check if there is a candidate which is actually a consistent set of strategies. (A less naive algorithm can be found in [6].)

In light of Corollary 4.3 and Lemma 4.4, success is guaranteed if the trees came from quasi-isometric groups. □

4.3. **Building Quasi-isometries.** A *P–set space* is the preimage in the model space of a P–set in the Bass-Serre tree. In Lemma 4.6 and Lemma 4.7 we construct quasi-isometries of P–set spaces. In Theorem 4.8 these are pieced together to give a quasi-isometry of tubular groups.

A P–set space looks like a graph of $\mathbb{Z}$'s, and the quasi-isometries we construct will be based on Whyte's construction of quasi-isometries between Baumslag-Solitar groups [13]. However, there are two additional complications. First, depth zero vertex spaces belong to several different P–set spaces. We must take care to build

quasi-isometries of P–set spaces which can later be pieced together into a quasi-isometry of tubular groups. Second, to make use of Theorem 4.5 we need to build according to a given extension, which puts type restrictions on which vertex spaces map to each other.

Let R be a P–set in $D\Gamma$. Let v_0 be some depth zero vertex of R. Height for other points of R can then be defined relative to v_0.

Put standard Euclidean coordinates on X_{v_0} in such a way that the edges of R glue onto X_{v_0} along lines of the form $t \times \mathbb{R}$.

The second coordinate can be projected over all of X_R, giving coordinates on X_R of the form tree $\times \mathbb{R}$, with the $\mathbb{R}$ factor scaled according $\exp(-\text{height})$. With these coordinates, an edge space or a positive depth vertex space is just $(t, \mathbb{R})$, for some t, and the length of $(t, [a, b])$ is $\exp(-h(t))|b - a|$.

If v is a depth zero vertex of R, parameterize the direction orthogonal to the lines of edge attachments so that the edge leading back to v_0 is at coordinate 0 and so that $d_{X_v} = \exp(-h(v))d_{\mathbb{R}\times\mathbb{R}}$.

Define a map $B\colon R \to \text{tree}$ by lifting to X_R and projecting to zero in the second coordinate. In $B(R)$ the depth zero vertices get "blown up" into lines. Edges of $B(R)$ contained in a depth zero vertex space will be called *horizontal*. Conversely, edges of $B(R)$ which cross an edge space will be called *vertical*.

A *horizontal component* of $B(R)$ is the blow up of some depth zero vertex. A *vertical component* is a connected component of the complement of the interiors of the horizontal edges.

The next lemma gives a quasi-isometry of a P–set space for a P–set of bounded height change. The various parameters just say that this is a quasi-isometry, built according to an extension $\mathcal{E}$, starting from a previously defined map on one of the depth zero vertex spaces.

Lemma 4.6. *Let $G_1 = \pi_1(\Gamma_1)$ and $G_2 = \pi_1(\Gamma_2)$ be two tubular groups. Let $R \subset D\Gamma_1$ and $S \subset D\Gamma_2$ be P–sets of bounded height change. Let $x_0 \in R$ and $y_0 \in S$ be depth zero vertices. Let $\mathcal{E} = (e_{ij})$ be an extension for $([R], [S])$. Let α and β be arbitrary positive real numbers. Then there is a quasi-isometry $\Phi = \Phi^0_{\mathcal{E},(R,x_0,\alpha),(S,y_0,\beta)}\colon X_R \to Y_S$ with the following properties:*

1. *$\Phi(X_{x_0}) = Y_{y_0}$*
2. *Φ induces a bijection, $\Phi_\#\colon \mathcal{V}_0R \to \mathcal{V}_0S$.*
3. *Up to isometry, $\Phi|_{X_v}$ is a homothety with expansion by $\frac{\beta}{\alpha}\exp(h(v) - h(\Phi_\#(v)))$.*
4. *Excluding x_0 and y_0, there exist vertices of type $\{[R], i\}$ mapping to vertices of type $\{[S], j\}$ $\iff$ e_{ij} is non-zero.*
5. *Φ is a bilipschitz bijection $(\mathcal{V}_0X_R, d_{X_R}|_{\mathcal{V}_0X_R}) \to (\mathcal{V}_0Y_S, d_{Y_S}|_{\mathcal{V}_0Y_S})$.*

Proof. Pick some J such that $\frac{J}{2}$ is a bound for the height change between two vertices of R or of S.

For any $x \in R$, let $h(x) = h(x_0, x)$, and similarly for $y \in S$ with respect to y_0.

Let x be a depth zero vertex of R. Let e be an edge of R incident to x. Since $\text{Stab}_{G_1}(e)$ has infinite index in $\text{Stab}_{G_1}(x)$, the orbit of e by $\text{Stab}_{G_1}(x) \cap \text{Stab}_{G_1}(R)$ contains infinitely many other edges incident to x. The edge spaces for these edges glue on to X_x along parallel lines, and the distance between such lines can be bounded in terms of $[x]$, $[R]$, and $[e]$.

Pick some L such that for each $P = R$, S, and every $v \in \mathcal{V}_0P$, any open interval of length L in $B(v)$ has at least two incident vertical edges from each

$\mathrm{Stab}_G(v) \cap \mathrm{Stab}_G(P)$ equivalence class of edge incident to v, and at least three total incident vertical edges.

Choose coordinates for X_R as discussed above. Choose coordinates for Y_S in a similar fashion, with the following provision. If we already have a map $X_{x_0} \to Y_{y_0}$, choose the coordinates of Y_{y_0} so that the origin of Y_{y_0} is the image of the origin of X_{x_0} and so that map preserves the orientation of the second coordinate.

For every $v \in \mathcal{V}_0 P$, do the following: Define the basepoint ξ_v of $B(v)$ to be the point of $B(v)$ with coordinate zero. Slide any edge incident to the open interval

$$\left(-\frac{1}{2}\exp(h(v))L, \frac{1}{2}\exp(h(v))L)\right)$$

in $B(v)$ to 0. Note that for each v this interval is of length L.

These sliding operations change distances between points in different horizontal components, but preserve distance within a fixed horizontal component.

For the vertices of the vertical components which are points of intersection with horizontal components, the *type* of the vertex is just the type of the vertex in $D\Gamma$ corresponding to that horizontal component. After sliding, the new vertical components are composed of unions of the original vertical components. In particular, the new vertical components containing ξ_{x_0} and ξ_{y_0} are infinite, bounded valence trees; call them V and W, respectively.

The image of V in Γ_1 is the same as the image of R in Γ_1. Therefore, the set of vertices of V of any particular vertex type are coarsely dense in V. The diameter of the image of R in Γ_1 provides a coarseness constant.

Similar statements are true for W as well. Let δ be the greater of the diameters of R in Γ_1 and S in Γ_2.

Every vertex of V and W is distance at most δ from a vertex of valence at least three. We will say V and W are δ-*bushy*. Also, the valence of V and W is bounded above.

For convenience we will assume that the extension $\mathcal{E}$ gives a bijection from vertex types of R to vertex types of S. To arrange this, note that the set of vertices of V of a particular type can be subdivided in to a finite number of subsets, each of which is also dense in V, and similarly for W.

Bounded valence bushy trees are quasi-isometric with quasi-isometry constants depending only on the valence bound and the bushiness constant. Therefore, there is a quasi-isometry $f\colon V \to W$. Furthermore, using non-amenability and Hall's Selection Theorem, the quasi-isometry f can be changed a bounded amount so that it is a bilipschitz bijection of depth zero vertices and respects the partitions of vertices by type, cf [6, Lemma 4]. The bilipschitz constant depends only on the valence bounds, bushiness constants, density constants, and number of vertex types.

Let ϕ be this bilipschitz bijection, and let M_x be the bilipschitz constant.

This process will be called extension along a vertical component. Note that every depth zero vertex in V or W is the basepoint of its horizontal component.

Suppose extension along a vertical component identifies ξ_x to ξ_y.

Let

$$N_x = \max\{L\exp(h(x)), L\frac{\alpha}{\beta}\exp(h(y))\}.$$

For $n = 1, 2, \dots$, consider the half-open intervals

$$\left[\frac{1}{2}\exp(h(x))L + (n-1)N_x, \frac{1}{2}\exp(h(x)) + nN_x\right) \subset B(x)$$
$$\left(-\frac{1}{2}\exp(h(x))L - nN_x, -\frac{1}{2}\exp(h(x)) - (n-1)N_x\right] \subset B(x)$$
$$\left[\frac{1}{2}\exp(h(y))L + (n-1)\frac{\beta}{\alpha}N_x, \frac{1}{2}\exp(h(y)) + n\frac{\beta}{\alpha}N_x\right) \subset B(y)$$
$$\left(-\frac{1}{2}\exp(h(y))L - n\frac{\beta}{\alpha}N_x, -\frac{1}{2}\exp(h(y)) - (n-1)\frac{\beta}{\alpha}N_x\right] \subset B(y)$$

Each of these intervals has length at least L, so there are at least three vertical edges incident to each interval. On the other hand, the length of the intervals, is bounded above by $N = L\exp(J)\max\{\frac{\alpha}{\beta}, \frac{\beta}{\alpha}\}$.

In $B(x)$, slide all incident vertical edges to the closed endpoint of the interval to which they attach. In $B(y)$, slide incident vertical edges in the nth right open interval to the point $\frac{\beta}{\alpha}(\frac{1}{2}\exp(h(x))L + (n-1)N_x)$. Perform similar operations for the left open intervals of $B(y)$.

Extend ϕ by $\phi|_{B(x)}\colon B(x) \to B(y)\colon r \mapsto \pm\frac{\beta}{\alpha}r$. The choice of orientation here is determined by the element e_{ij} associated to x, y. The edge sliding matches up the points to which vertical edges attach, and each of these will be the basepoint of a new vertical component to extend along.

Alternate extending along collections of vertical and horizontal components to build ϕ.

We make the following observations about this construction:

(1) Every vertical edge has at most one endpoint which slides.
(2) No edge slides more than once.
(3) Sliding an edge only changes distances between points separated by the edge.
(4) No edge slides more than distance N.

Distance in $B(\mathcal{V}_0 R)$ and $B(\mathcal{V}_0 S)$ therefore change by at most a multiplicative factor of $1 + N$. Extension along horizontal components changes distance by at most a multiplicative factor of $\frac{\beta}{\alpha}\exp J$. Vertical components have valence bounded by N and information from X and Y, so there is a uniform bound M on the bilipschitz constants M_x. Thus, $(1+N)^2 M\frac{\beta}{\alpha}\exp(J)$ give a bilipschitz constant for $\phi\colon B(\mathcal{V}_0 R) \to B(\mathcal{V}_0 S)$.

Define $\Phi\colon B(R) \times \mathbb{R} \to B(S) \times \mathbb{R}$ by $(t, u) \mapsto (\phi(t), \frac{\beta}{\alpha}u)$.

Geodesics in X_R and Y_S can be approximated within bounded multiplicative error by paths in which only one coordinate changes at a time, so Φ is bilipschitz on $\mathcal{V}_0 X_R$.

The union of vertex spaces is dense in a P–set space, so Φ gives a quasi-isometry of P–set spaces. □

We have a similar result for P–sets of unbounded height change.

Lemma 4.7. *Let $G_1 = \pi_1(\Gamma_1)$ and $G_2 = \pi_1(\Gamma_2)$ be two tubular groups. Let $R \subset D\Gamma_1$ and $S \subset D\Gamma_2$ be P–sets of unbounded height change. Let $x_0 \in R$ and $y_0 \in S$ be depth zero vertices. Let $\mathcal{E} = (e_{ij})$ be an extension for $([R], [S])$. Let*

α and β be arbitrary positive real numbers. Then there is a quasi-isometry $\Phi = \Phi^{\infty}_{\mathcal{E},(R,x_0,\alpha),(S,y_0,\beta,)}\colon X_R \to Y_S$ with the following properties:

1. *$\Phi(X_{x_0}) = Y_{y_0}$, and this map is, up to isometry, just a homothety with expansion by $\frac{\beta}{\alpha}$.*
2. *Φ induces a bijection, $\Phi_{\#}\colon \mathcal{V}_0R \to \mathcal{V}_0S$.*
3. *Up to isometry, and for $v \neq x_0$, $\Phi|_{X_v}$ is a homothety with uniformly bounded expansion factor.*
4. *Excluding x_0 and y_0, there exist vertices of type $\{[R], i\}$ mapping to vertices of type $\{[S], j\}$ iff e_{ij} is non-zero.*
5. *Φ is a bilipschitz bijection $(\mathcal{V}_0X_R, d_{X_R}|_{\mathcal{V}_0X_R}) \to (\mathcal{V}_0Y_S, d_{Y_S}|_{\mathcal{V}_0Y_S})$.*

Proof. The proof differs from the proof of Lemma 4.6 only in the how we extend along vertical components.

For $i = 1, 2$, consider the projection of the appropriate P–set to the graph Γ_i. Let μ_i be the largest height change that occurs across a single edge in the projection. The edge sliding is set up so that there are coarsely height preserving quasi-isometries between the resulting vertical components. Furthermore, the height error can be bounded in terms of the projections to the $Gamma_i$ [6]. Let K be such a bound; K plays the role in this case that the constant J played in the bounded height change case.

Perform the edge sliding as in the previous Lemma. Suppose V and W are vertical components based at $v_0 \in B(x)$ and $w_0 \in B(y)$, respectively. Notice that the valence of V at v_0 may depend on $\frac{\beta}{\alpha}$, since v_0 may not be equal to ξ_x. However, every other vertex in $V \cap B(\mathcal{V}_0R)$ and $W \cap B(\mathcal{V}_0S)$ is the basepoint of its vertical component, so the valence of the trees at these points is bounded independently of $\frac{\beta}{\alpha}$. It is possible to "disperse" the extra edges at v_0 by a height and type preserving bilipschitz bijection $V \cap B(\mathcal{V}_0R)$ to a new tree V'. In the same way replace W by a new tree W'.

There is some vertex ξ_w in W at bounded distance from ξ_{w_0} such that $|h(\xi_{v_0}, \xi_{w_0}) + \ln\frac{\beta}{\alpha}| \leq \mu_2$. Let ϕ be a $K + \mu_2$-coarsely height preserving quasi-isometry between V' and W' that maps v_0 to ξ_w. As in the previous Lemma, ϕ, can then be changed a bounded amount to give a bijection ϕ' of depth zero vertices which respects the partitions by type. Moreover, the amount ϕ needs to be changed is independent of $\frac{\beta}{\alpha}$, so the height error is still independent of $\frac{\beta}{\alpha}$.

The map $V \to V' \to W' \to W$ is than a bilipschitz bijection $V \cap B(\mathcal{V}_0R) \to W \cap B(\mathcal{V}_0S)$. Finally, we adjust this map to ensure v_0 maps to w_0. The result is the map we will use to extend along vertical components. Note that while the bilipschitz constants depend on $\frac{\beta}{\alpha}$, the height error at points other than v_0 is bounded independently of $\frac{\beta}{\alpha}$. □

Remark. Note the important difference in these two Lemmas. In both cases we start by identifying x_0 to y_0. The parameters α and β provide an initial height error; think of $E = \ln(\frac{\beta}{\alpha})$ as the height error at x_0.

In the bounded case, the height error of the other vertices is in $[E - J, E + J]$.

In the unbounded case, E influences the quasi-isometry constants, but the height error of the other vertices is independent of E.

The following Theorem shows that a consistent set of strategies can be used to build a quasi-isometry between tubular groups. Lemma 4.6 and Lemma 4.7

provide the basic building blocks, and the quasi-isometry constants are controlled by controlling height error. Two-line, depth zero vertices and P–sets of unbounded height change are flexible enough that height error will not be an issue. The danger of compounding height error comes from P–sets of bounded height change connected by depth zero vertices with at least three lines, but these height errors are controlled by the set of strategies.

Theorem 4.8. *For $i = 1, 2$, let $G_i = \pi_1(\Gamma_i)$ be a tubular group and T_i its tree of P–sets. If there is a consistent set of strategies for T_1 and T_2, then G_1 and G_2 are quasi-isometric.*

Proof. Let $\mathcal{G}$ be a consistent set of strategies for T_1 and T_2. We build a quasi-isometry $\Phi\colon X \to Y$ and an allowable tree isomorphism $\phi\colon T_1 \to T_2$.

Let $\mathcal{M}_i$ be the matches of P–sets of bounded height change occurring in $\mathcal{G}$, and let $\{U_i\}$ and $\{L_i\}$ be height error bounds from the consistency check.

Choose a uniform K such that, for any P–sets $R \in T_1$ and $S \in T_2$ of unbounded height change, there is a quasi-isometry $\Phi^\infty\colon X_R \to Y_S$ as in Lemma 4.7 with height error bounded by K.

In $D\Gamma_1$ and $D\Gamma_2$ there are finitely many equivalence classes of P–sets of bounded height change, so there is some maximum height change, $\frac{J}{2}$, that can occur between depth zero vertices in such a P–set.

Assume the U_i are greater than $K + J$ and the L_i are less than $-J - K$.

Pick any P–set vertices R_0 of T_1 and S_0 of T_2 such that $([R_0], [S_0])$ is the match of a vertex of $\mathcal{G}$. Set $\phi(R_0) = S_0$.

Base Case 1. If R_0 is of bounded height change, pick a depth zero vertex $x_0 \in R_0$ of maximal height. Declare x_0 to be the basepoint, so x_0 has height zero and all other heights are determined relative to x_0.

Also assume that $\mathcal{M}_1 = ([R_0], [S_0])$.

Consult $\mathcal{G}$ for a positive strategy, $\mathcal{S}$, with an extension, $\mathcal{E}$, for $([R_0], [S_0])$. If x_0 is of type $\{[R_0], i\}$ find some j such that the i, j entry of $\mathcal{E}$ is nonzero and choose a depth zero vertex $y_0 \in S_0$ of type $\{[S_0], j\}$.

Declare y_0 to be height zero, and define heights in S_0 relative to y_0.

With these choices, $h(R_0) = 0$, and $h(S_0) \geq 0$, so the height error of R is in $[-\frac{J}{2}, 0]$.

Let $\Phi_{R_0} = \Phi^0_{\mathcal{E},(R_0,x_0,1),(S_0,y_0,1)}$ from Lemma 4.6.

Define $\Phi|_{X_{R_0}} = \Phi_{R_0}$.

The map Φ_{R_0} induces a map $(\Phi_{R_0})_\#\colon N_{\frac{1}{2}}(R_0) \to N_{\frac{1}{2}}(S_0)$ which is a tree isomorphism. Extend ϕ from R_0 according to $\mathcal{E}$, using the bijection of depth zero vertices furnished by $(\Phi_{R_0})_\#$.

The height errors of the depth zero vertices of R_0 are in $[-J, \frac{J}{2}]$.

Base Case 2. If R_0 is a P–set of unbounded height change pick a depth zero vertex, x_0, and declare its height to be zero. Height relative to x_0 determines heights for all the other depth zero vertices of R_0.

Consult $\mathcal{G}$ for a strategy, $\mathcal{S}$, with an extension, $\mathcal{E}$, for $([R], [S])$. If x_0 is of type $\{[R_0], i\}$ find some j such that the i, j entry of $\mathcal{E}$ is nonzero, and choose a depth zero vertex $y_0 \in S_0$ of type $\{[S_0], j\}$. Declare the height of y_0 to be zero, and define height in S_0 relative to y_0.

Let $\Phi_{R_0} = \Phi^\infty_{\mathcal{E},(R_0,x_0,1),(S_0,y_0,1)}$ from Lemma 4.7.

Define $\Phi|_{X_{R_0}} = \Phi_{R_0}$.

Extend ϕ from R_0 according to $\mathcal{E}$, using the bijection of depth zero vertices furnished by $(\Phi_{R_0})_\#$.

By construction, the height errors of the depth zero vertices of R_0 are in $[-K, K]$.
Induction Steps. Suppose Φ has been defined for a P–set space X_R, and ϕ has been defined on $N_1(R)$. Let R' be a P–set vertex at distance 1 from R, and let v be the depth zero vertex joining R to R'.

There are a number of situations to consider, depending on whether R and R' are of bounded or unbounded height change, and whether v has 2 lines or more than 2 lines.

In each of these cases, we give a quasi-isometry $\Phi_{X_{R'}}$ from the P–set space of R' to the P–set space of $\phi(R')$ according to some extension $\mathcal{E}$.

To finish each case, define $\Phi|_{X_{R'}} = \Phi_{R'}$, and extend ϕ from R' according to $\mathcal{E}$, using the bijection of depth zero vertices furnished by $(\Phi_{R'})_\#$.
v has more than 2 lines. In these cases, R and R' are in the same rigid component, so height in R' is already defined. The same goes for $\phi(R)$ and $\phi(R')$.
R and R' of bounded height change. Suppose $([R], [\phi(R)]) = \mathcal{M}_i$, and suppose $([R'], [\phi(R')]) = \mathcal{M}_j$. If the height error of R was in $[L_i, U_i]$, then the height error of R' is in $[L_j, U_j]$.

If the height error of R' is positive, let $\mathcal{S}$ be the negative strategy for $\mathcal{M}_j$ in $\mathcal{G}$. If the height error of R' is negative, let $\mathcal{S}$ be the positive strategy for $\mathcal{M}_j$ in $\mathcal{G}$.

The strategy $\mathcal{S}$ has an extension, $\mathcal{E}$, for $\mathcal{M}_j$.

Let $\alpha = \exp(-h(v))$; let $\beta = \exp(-h(\phi(v)))$.

Let $\Phi_{R'} = \Phi^0_{\mathcal{E},(R',v,\alpha),(\phi(R'),\phi(v),\beta)}$ from Lemma 4.6.

The height errors of depth zero vertices of R' are in $[L_j - J, U_j + J]$.
R of unbounded height change, R' of bounded height change. Since R was of unbounded height change, we can assume that the height error of v is in $[-K, K]$. Suppose $([R'], [\phi(R')]) = \mathcal{M}_j$. The height error of R' is in $[-K-J, K+J] \subset [L_j, U_j]$.

Proceed as in the previous case.

The height errors of depth zero vertices of R' are in $[-K - J, K + J]$.
R' of unbounded height change, both cases. Let $\mathcal{E}$ be the extension given by the strategy for $([R'], [\phi(R')])$ in $\mathcal{G}$.

Let $\alpha = \exp(-h(v))$; let $\beta = \exp(-h(\phi(v)))$.

Let $\Phi_{R'} = \Phi^0_{\mathcal{E},(R',v,\alpha),(\phi(R'),\phi(v),\beta)}$ from Lemma 4.7.

By construction, the height errors of depth zero vertices of R' (other than v) are in $[-K, K]$.
v has 2 lines. In these cases, R and R' are in different rigid components. Define height in R' relative to v and define height in $\phi(R')$ relative to $\phi(v)$.
R' of bounded height change. The P–sets R' and $\phi(R')$ have heights defined relative to v and $\phi(v)$, respectively, and the height error of R' is in $[-J, J]$. If the height error of R' is positive, let $\mathcal{S}$ be the negative strategy of $\mathcal{G}$ for $([R'], [\phi(R')])$. If the height error of R' is negative, choose the positive strategy. Let $\mathcal{E}$ be the extension given by $\mathcal{S}$.

Let $\Phi_{R'} = \Phi^0_{\mathcal{E},(R',v,1),(\phi(R'),\phi(v),1)}$.

The height errors of depth zero vertices of R' are in $[-J, J]$.
R' of unbounded height change. Let $\mathcal{E}$ be the extension associated to the strategy for $(R', \phi(R'))$ given in $\mathcal{G}$.

Let $\Phi_{R'} = \Phi^\infty_{\mathcal{E},(R',v,1),(\phi(R'),\phi(v),1)}$.

By construction, the height errors of the depth zero vertices of R' are in $[-K, K]$.

This completes the induction steps.

The various maps Φ_R were constructed so that they agree when they overlap on any depth zero vertex spaces of at least three lines. They may not agree on two-line vertex spaces, but this can be fixed, since two-line patterns are not rigid.

Suppose v is a two-line vertex joining P–sets R and R'. Suppose $\Phi_R|_{X_v}$ is expansion by a factor C, and $\Phi_{R'}|_{X_v}$ is expansion by a factor C'. Change these maps so that they both expand by a factor of C along the lines where the R' edges glue on, and expand by a factor of C' along the lines where the R edges glue on. This does not change the maps that we had on $B(v) \subset B(R)$ and $B(v) \subset B(R')$.

The map Φ is built from various pieces Φ_R. The Φ_R are bilipschitz bijections on $\mathcal{V}_0 X_R$, and agree on vertex spaces where their domains intersect. We have arranged that the bilipschitz constants of Φ_R depend on $[R]$ and $[\phi(R)]$ and on height error. Tubular groups come from finite graphs of groups, so there are only finitely many equivalence classes of P–set. Thus, invariants of equivalence classes of P–sets can be uniformly bounded. Height error is bounded by $J + \max\{U_i\} \cup \{|L_i|\}$.

The map Φ is thus a bilipschitz bijection $\mathcal{V}_0 X \to \mathcal{V}_0 Y$. The set $\mathcal{V}_0 X$ is dense in X, and $\mathcal{V}_0 Y$ is dense in Y, so $\Phi\colon X \to Y$ is a quasi-isometry. □

Theorem 4.9. *For $i = 1, 2$, let $G_i = \pi_1(\Gamma_i)$ be tubular groups. There is an algorithm which in finite time determines whether or not G_1 and G_2 are quasi-isometric.*

Proof. Run the algorithm from Theorem 4.5. This algorithm halts in finite time. If the groups are quasi-isometric then Corollary 4.3 and Lemma 4.4 guarantee that the algorithm succeeds and produces a consistent set of strategies. Conversely, Theorem 4.8 builds a quasi-isometry from a consistent set of strategies, so the algorithm succeeds only if the groups are quasi-isometric. □

5. Consequences and Examples

Suppose G is a tubular group with at least 3 lines in every vertex and having bounded height change in every P–set. Then there is a notion of height change for every edge in the tree of P–sets.

If γ is a geodesic ray in T_G, γ *has coarse slope* m, $\operatorname{slope}(\gamma) = m$, if there exists a $C > 0$ such that for all t,

$$tm - C \leq h(\gamma(0), \gamma(t)) \leq tm + C$$

Proposition 5.1. *There is some $m_G = \max_{\gamma \subset T_G} \operatorname{slope}(\gamma)$, and the set of slopes of geodesic rays of T_G is dense in the interval $[-m_G, m_G]$.*

Proof. Let γ be a geodesic ray in T_G. Consider the image of γ in the graph which is the quotient of T_G by the group action. This graph is finite, so the image of γ can be written as some initial segment of bounded length followed by loops in the graph. The initial segment contributes a bounded amount to height, so it can be discarded.

There are only finitely many simple closed curves in a finite graph, so pick one, α, of maximal slope. It is not hard to see that any loop can be reduced to a simple closed curve without reducing slope. Therefore $\operatorname{slope}(\gamma) \leq \operatorname{slope}(\alpha)$. Then $m_G = \operatorname{slope}(\alpha)$.

Loops which are homotopically trivial have 0 height change, so we can get a geodesic ray in T_G of slope arbitrarily close to any $m \in [-m_G, m_G]$ by concatenating lifts of $\pm\alpha$ with lifts of trivial loops. □

Corollary 5.2. *There is a maximum coarse slope in each rigid component of T_G which contains no P–set of unbounded height change.*

Suppose G_1 and G_2 are tubular groups with at least three lines in every vertex and having bounded height change in every P–set. Also, suppose that G_1 and G_2 have the same sets of equivalence classes of edge patterns, and that each of these patterns has a unique (up to isometry) symmetric representative. This occurs, for instance, if every vertex has 3 or 4 lines.

Suppose $m_{G_1} > m_{G_2}$, and γ is a geodesic ray of slope m_{G_1} in T_{G_1}. Under a coarsely height preserving tree isomorphism, the image of γ will still be a geodesic ray of slope m_{G_1}, contradicting maximality of m_{G_2}. This means m_G is a quasi-isometry invariant of G.

When vertices of G have edge patterns without a unique symmetric representative, the value of m_G depends on the choice of representatives.

Having the same maximum slope is a necessary, but not sufficient condition for determining if the groups are quasi-isometric, as illustrated by the following example.

Note. In diagrams of trees of P–sets, P–sets are represented by large open circles labeled by a representative of the equivalence class of P–set. One adjacent vertex is added for each P–set-stabilizer equivalence class of depth zero vertex contained in the P–set, and these are labeled by vertex types with respect to the more heavily shaded P–set in the center of the diagram.

Example 5.3. Suppose G_1 and G_2 are groups as in Figure 1. Each G_i has only one equivalence class of P–set, $[C_i]$, and each P–set has three vertex types.

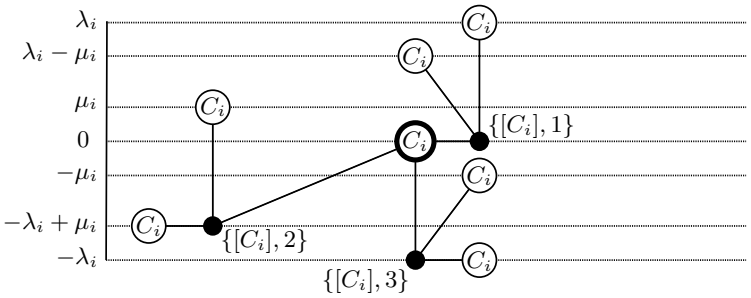

FIGURE 2. Vertex types and relative heights for $[C_i]$.

Assume that $\lambda_i \geq \mu_i \geq 0$, and that $\lambda_i > 0$.

Figure 2 summarizes the setup.

The maximum slope in T_i is λ_i, so assume that $\lambda_1 = \lambda_2$.

First we attempt to build an extension $\mathcal{E} = (e_{jk})$ that gives a positive strategy.

To build a consistent set of strategies we will need a positive strategy that gives all non-negative height errors. The largest negative height change that occurs is $-\lambda$ in both trees, so we must identify these. To accomplish this, e_{33} must be non-zero. Furthermore, if we choose this entry to identify the P–sets at height $-\lambda$, we are

forced to identify P–sets at height $-\mu_1$ with P–sets of height $-\mu_2$. Thus, we need $\mu_1 \leq \mu_2$.

We get the reverse inequality by considering a negative strategy. We need a negative strategy that gives all non-positive height errors. The largest positive height change that occurs is λ in both trees, so we must identify these. This forces an identification of P–sets at height $\lambda - \mu_1$ with P–sets of height $\lambda - \mu_2$, so we need $\mu_1 \geq \mu_2$.

Thus, if $\mu_1 \neq \mu_2$, the two groups can not be quasi-isometric.

When $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$ the obvious strategy produces zero height error, so the groups are quasi-isometric.

Example 5.4. Consider two tubular groups all of whose depth zero vertices have two lines. The only obstruction to building a quasi-isometry in this case is that P–sets of bounded height change must be matched to P–sets of bounded height change, and similarly for P–sets of unbounded height change.

In particular, we recover the following result of Behrstock and Neumann [1]:

Corollary 5.5. *Any two Right Angled Artin Groups whose defining graphs are trees of diameter at least three are quasi-isometric.*

This follows since such groups have graph of groups decompositions as tubular groups where all depth zero vertices have two lines and all P–sets are of bounded height change.

Example 5.6. Let $G_1 = \pi_1(\Gamma_1)$ be a one torus group as in Example 5.3 with height changes $\lambda > \mu > 0$ across the two edges.

Let $G_2 = \pi_1(\Gamma_2)$ be a tubular groups whose underlying graph is shown in Figure 3. In this diagram the labels are height changes and an arc joining edges incident to a common vertex indicates that the corresponding edge strips glue to the vertex space along parallel lines.

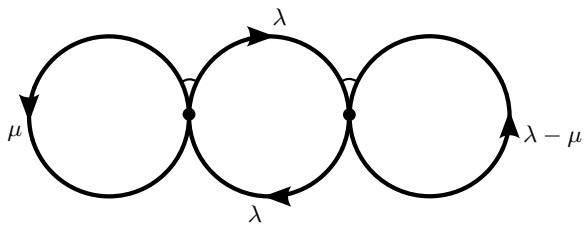

FIGURE 3. A graph for a two torus group.

The Bass-Serre tree, $D\Gamma_2$, has two equivalence classes of P-sets. One class, $[A]$, consists of P-sets which project in Γ_2 to the union of the three edges connected by arcs. The other, $[B]$, consists of P-sets which project to the remaining edge in Γ_2.

Figure 4 and Figure 5 define the vertex types for vertices in A and B, respectively.

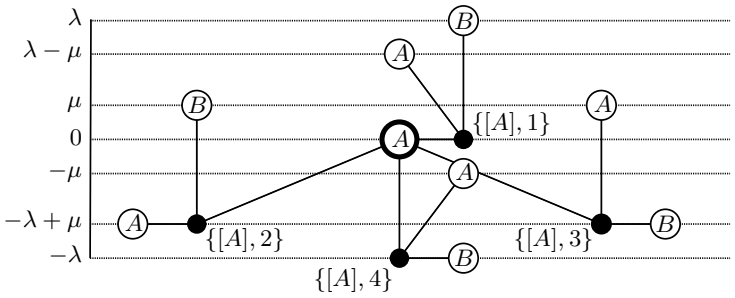

FIGURE 4. Vertex types and relative heights for $[A]$.

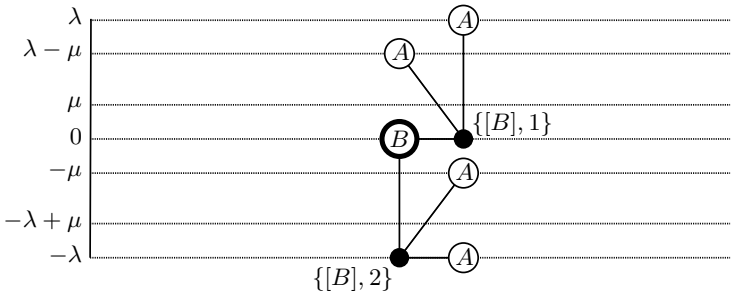

FIGURE 5. Vertex types and relative heights for $[B]$.

For $([C],[A])$, consider the extension $\mathcal{E}_A = (e_{ij})$ defined as follows:

$$e_{11} = \left\{\begin{array}{l}\{[C],1\}\mapsto\{[A],1\}\\ \{[C],2\}\mapsto\{[A],2\}\\ \{[C],3\}\mapsto\{[B],2\}\end{array}\right\} \quad e_{22} = \left\{\begin{array}{l}\{[C],1\}\mapsto\{[A],1\}\\ \{[C],2\}\mapsto\{[A],2\}\\ \{[C],3\}\mapsto\{[B],2\}\end{array}\right\}$$

$$e_{23} = \left\{\begin{array}{l}\{[C],1\}\mapsto\{[B],2\}\\ \{[C],2\}\mapsto\{[A],3\}\\ \{[C],3\}\mapsto\{[A],4\}\end{array}\right\} \quad e_{34} = \left\{\begin{array}{l}\{[C],1\}\mapsto\{[B],1\}\\ \{[C],2\}\mapsto\{[A],3\}\\ \{[C],3\}\mapsto\{[A],4\}\end{array}\right\}$$

$$e_{ij} = \quad 0 \text{ for all other } ij$$

This extension says that given P–sets C and A, we should split the vertices of type $\{[C],2\}$ with respect to C and map half of them to the vertices of type $\{[A],2\}$ with respect to A and the other half to vertices of type $\{[A],3\}$ with respect to A.

Figure 6 shows the result of extending by $\mathcal{E}_A$.

All the matches induced by this extension have height error 0, so the strategy $\mathcal{S}_A$ for $\mathcal{E}_A$ can be used as a positive and a negative strategy.

For $([C],[B])$ we define extensions $\mathcal{E}_B^+$ and $\mathcal{E}_B^-$, extension by which are depicted in Figure 7 and Figure 8, respectively.

The extension $\mathcal{E}_B^+$ induces matches $([C],[A])$ with height errors 0, μ, and 2μ, so it gives rise to a positive strategy.

This extension induces matches $([C],[A])$ with height errors 0, $-2(\lambda-\mu)$, $\lambda-\mu$, so it gives rise to a negative strategy.

These strategies give the set of strategies in Figure 9.

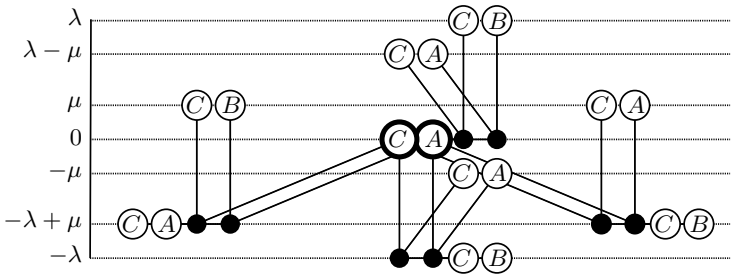

FIGURE 6. Extension by $\mathcal{E}_A$.

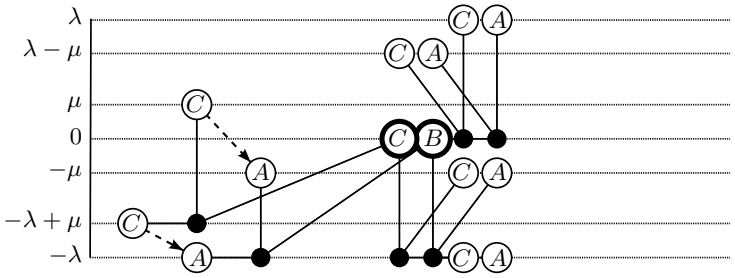

FIGURE 7. Extension by $\mathcal{E}_B^+$.

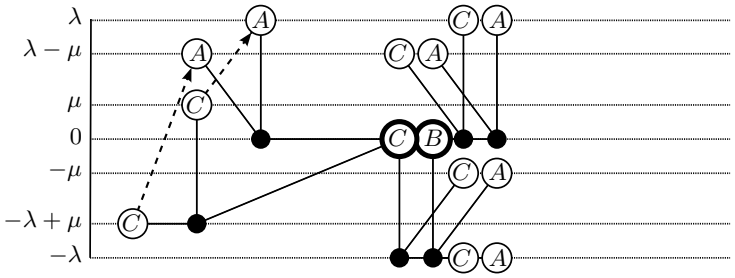

FIGURE 8. Extension by $\mathcal{E}_B^-$.

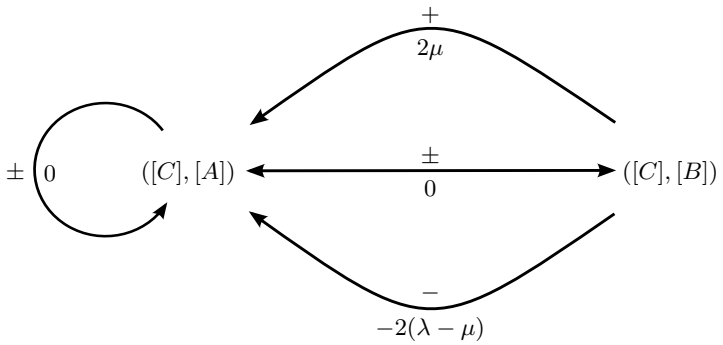

FIGURE 9. The set of strategies

This set of strategies gives rise to a system of inequalities:

$$\begin{array}{ccc} L_1 \leq M_1 \leq U_1 & M_1 + 0 \leq U_2 & U_2 - 0 \leq U_1 \\ L_2 \leq M_2 \leq U_2 & L_1 + 0 \geq L_2 & M_2 - 0 \geq L_1 \\ U_1 + 0 \leq U_2 & M_2 + 2\mu \leq U_1 & U_2 - 2(\lambda - \mu) \leq U_1 \\ M_1 + 0 \geq L_2 & L_2 + 2\mu \geq L_1 & M_2 - 2(\lambda - \mu) \geq L_1 \end{array}$$

This system has solutions, for instance:

$$\begin{aligned} U_1 = U_2 &= 2\lambda \\ M_1 = M_2 &= 0 \\ L_1 = L_2 &= -2\lambda \end{aligned}$$

so groups of this form are quasi-isometric.

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Christopher H Cashen
Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
Chicago, IL 60607, USA
E-mail address: `ccashen@math.uic.edu`
URL: `http://www.math.uic.edu/~ccashen`