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On partial polynomial interpolation

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Abstract

The Alexander-Hirschowitz theorem says that a general collection of k double points in $\mathbf{P}^n$ imposes independent conditions on homogeneous polynomials of degree d with a well known list of exceptions. We generalize this theorem to arbitrary zero-dimensional schemes contained in a general union of double points. We work in the polynomial interpolation setting. In this framework our main result says that the affine space of polynomials of degree $\leq d$ in n variables, with assigned values of any number of general linear combinations of first partial derivatives, has the expected dimension if $d \neq 2$ with only five exceptional cases. If $d = 2$ the exceptional cases are fully described.

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1 Introduction

Let K be an infinite field. Let $R_{d,n} = K[x_1, \ldots, x_n]_d$ be the vector space of polynomials of degree $\leq d$ in n variables. Note that $\dim R_{d,n} = \binom{n+d}{d}$. Let $p_1, \ldots, p_k \in K^n$ be k general points and assume that over each of these points a general affine proper subspace $A_i \subset K^n \times K$ of dimension a_i is given. Assume that $a_1 \geq \ldots \geq a_k$. Let $\Gamma_f \subseteq K^n \times K$ be the graph of $f \in R_{d,n}$ and $T_{p_i}\Gamma_f$ be its tangent space at the point $(p_i, f(p_i))$. Note that $\dim T_{p_i}\Gamma_f = n$ for any i. Consider the conditions

$$A_i \subseteq T_{p_i}\Gamma_f, \text{ for } i = 1, \ldots, k. \tag{1}$$

When $a_i = 0$, the assumption (1) means that the value of f at p_i is assigned. When $a_i = n$, (1) means that the value of f at p_i and the values of all first partial derivatives of f at p_i are assigned. In the intermediate cases, (1) means that the value of f at p_i and the values of some linear combinations of first partial derivatives of f at p_i are assigned.

Consider now the affine space

$$V_{d,n}(p_1, \ldots, p_k, A_1, \ldots, A_k) = \{f \in R_{d,n} | A_i \subseteq T_{p_i}\Gamma_f, \quad i = 1, \ldots, k\}.$$

The polynomials in this space solve a partial polynomial interpolation problem. The condition (1) corresponds to (a_i+1) affine linear conditions on $R_{d,n}$. Our main result describes the codimension of the above affine space. Since the description is different for $d = 2$ and $d \neq 2$, we divide the result in two parts.

Theorem 1.1 *Let $d \neq 2$. For a general choice of points p_i and subspaces A_i, the affine space $V_{d,n}(p_1, \dots, p_k, A_1, \dots, A_k)$ has codimension in $R_{d,n}$ equal to*

$$\min\{\sum_{i=1}^{k}(a_i+1), \dim R_{d,n}\}$$

with the following list of exceptions

a) $n = 2, d = 4, k = 5, a_i = 2$ *for* $i = 1, \dots 5$.

b) $n = 3, d = 4, k = 9, a_i = 3$ *for* $i = 1, \dots 9$.

b′) $n = 3, d = 4, k = 9, a_i = 3$ *for* $i = 1, \dots 8$ *and* $a_9 = 2$.

c) $n = 4, d = 3, k = 7, a_i = 4$ *for* $i = 1, \dots 7$.

d) $n = 4, d = 4, k = 14, a_i = 4$ *for* $i = 1, \dots 14$

In particular when $\sum_{i=1}^{k}(a_i+1) = \binom{n+d}{d}$ *there is a unique polynomial f in $V_{d,n}(p_1, \dots, p_k, A_1, \dots A_k)$, with the above exceptions a), b'), c), d). In the exceptional cases the space $V_{d,n}(p_1, \dots, p_k, A_1, \dots A_k)$ is empty.*

The "general choice" assumption means that the points can be taken in a Zariski open set (i.e. outside the zero locus of a polynomial) and for each of these points the space A_i can be taken again in a Zariski open set. On the real numbers this assumption means that the choices can be done outside a set of measure zero. Our result is not constructive but it ensures that in the case $\sum_{i=1}^{k}(a_i+1) = \binom{n+d}{d}$ the linear system computing the interpolating polynomial with general data has a unique solution. Hence any algorithm solving linear systems can be successfully applied.

The case in which $a_i = n$ for all i was proved by Alexander and Hirschowitz in [AH1, AH2], see [BO] for a survey. The most notable exception is the case of seven points with seven tangent spaces for cubic polynomials in four variables, as in c). This example was known to classical algebraic geometers and it was rediscovered in the setting of numerical analysis in [Rei]. The case of curvilinear schemes was proved as a consequence of a more general result by [CG] on $\mathbf{P}^2$ and by [CM] in general.

The case $d = 1$ follows from elementary linear algebra. The case $n = 1$ is easy and well known: in this case the statement of Theorem 1.1 is true with the only requirement that the points p_i are distinct and the spaces A_i are not vertical, that is their projections $\pi(A_i)$ on K^n satisfy $\dim A_i = \dim \pi(A_i)$.

Assume now $d = 2$. We set $a_i = -1$ for $i > k$. For any $1 \leq i \leq n$ we denote

$$\delta_{a_1,\dots,a_k}(i) = \max\{0, \sum_{j=1}^{i} a_j - \sum_{j=1}^{i}(n+1-j)\}.$$

Theorem 1.2 *For a general choice of points p_i and subspaces A_i, the affine space $V_{2,n}(p_1, \dots, p_k, A_1, \dots A_k)$ has codimension in $R_{2,n}$ equal to*

$$\min\{\sum_{i=1}^{k}(a_i+1), \dim R_{2,n}\}$$

if and only if one of the following conditions takes place:

1. *either $\delta_{a_1,\dots,a_k}(i) = 0$ for all $1 \leq i \leq n$;*
2. *or $\sum_i (a_i+1) \geq \binom{n+2}{2} + \max\{\delta_{a_1,\dots,a_k}(i) : 1 \leq i \leq n\}$.*

In particular when $\sum_{i=1}^{k}(a_i+1)=\binom{n+2}{2}$ *there is a unique polynomial* f *in* $V_{2,n}(p_1,\dots,p_k,A_1,\dots A_k)$ *if and only if, for any* $1\le i\le n$,

$$\sum_{j=1}^{i}a_j\le\sum_{j=1}^{i}(n+1-j).$$

The first nontrivial example which explains Theorem 1.2 is the following. Consider $k=2$ and $(a_1,a_2)=(n,n)$. Then the affine space $V_{2,n}(p_1,p_2,A_1,A_2)$ is given by quadratic polynomials with assigned tangent spaces A_1, A_2 at two points p_1, p_2. This space is not empty if and only if the intersection space $A_1\cap A_2$ is not empty and its projection on K^n contains the midpoint of p_1p_2, which is a codimension one condition. In order to prove this fact restrict to the line through p_1 and p_2 and use a well known property of the tangent lines to the parabola. In this case $\delta_{n,n}(i)=\begin{cases}0 & i\neq 1\\ 1 & i=1\end{cases}$ and the two conditions of Theorem 1.2 are not satisfied. In Section 3 we will explain these two conditions in graphical terms.

Let $\pi(A_i)$ be the projection of A_i on K^n. For $i=1,\dots,k$ we consider the ideal

$$I_i=\{f\in K[x_1,\dots,x_n]|f(p_i)+\sum_{j=1}^{n}(x_j-(p_i)_j)\frac{\partial f}{\partial x_j}(p_i)=0 \text{ for any } x\in\pi(A_i)\}$$

Notice that we have $m_{p_i}^2\subseteq I_i\subseteq m_{p_i}$ and the ring $K[x_1,\dots,x_n]/I_i$ corresponds to a zero-dimensional scheme ξ_i of length a_i+1, supported at p_i and contained in the double point p_i^2. When $V_{d,n}(p_1,\dots,p_k,A_1,\dots,A_k)$ is not empty, its associated vector space (that is its translate containing the origin) consists of the hypersurfaces of degree d through $\xi_1,\dots,\xi_k$. Moreover, when this vector space has the expected dimension, it follows that $V_{d,n}(p_1,\dots,p_k,A_1,\dots,A_k)$ has the expected dimension too.

The space K^n can be embedded in the projective space $\mathbf{P}^n$. Since the choice of points is general, we can always avoid the "hyperplane at infinity". In order to prove the above two theorems, we will reformulate them in the projective language of hypersurfaces of degree d through zero-dimensional schemes. More precisely we refer to Theorem 3.2 for $d=2$, Theorem 4.1 for $d=3$ and Theorem 5.6 for $d\ge 4$. This reformulation is convenient mostly to rely on the wide existing literature on the subject. In this setting Alexander and Hirschowitz proved that a general collection of double points imposes independent conditions on the hypersurfaces of degree d (with the known exceptions) and our result generalizes to a general zero-dimensional scheme contained in a union of double points. It is possible to degenerate such a scheme to a union of double points only in few cases, in such cases of course our result is trivial from [AH1].

Anyway we underline that no advanced technique in algebraic geometry is involved in the proof, which remains essentially at the level of linear algebra. This explain our assumption on the base field, which is assumed to be infinite (and not necessarily algebraically closed), like in [AH1]. For finite fields the genericity assumption is meaningless.

Our proof of Theorem 5.6, and hence of Theorem 1.1, is by induction on n and d. Since it is enough to find a particular zero-dimensional scheme which imposes independent condition on hypersurfaces of degree d, we specialize some of the points on a hyperplane, following a technique which goes back to Terracini. We need a generalization of the Horace method, like in [AH1]. The case of cubics, which is the starting point of the induction, is proved by generalizing the approach

of [BO], where we restricted to a codimension three linear subspace. This case is the crucial step which allows to prove Theorem 1.1. While the induction argument works quite smoothly for $n, d \gg 0$, it is painful to cover many of the initial cases.

A further remark is necessary. In [AH1, BO] the result about the independence of double points was shown to be equivalent, through Terracini lemma, to a statement about the dimension of higher secant varieties and the Waring problem for polynomials. Here the assumption that K is algebraically closed of zero characteristic is necessary to translate safely the results, see also Theorem 6.1 and Remark 6.3 in [IK]. For example, on the real numbers, the closure in the euclidean topology of the locus of secants to the twisted cubic is a semi-algebraic set, corresponding to the cubic polynomials which have no three distinct real roots, which is defined by the condition that the discriminant is nonpositive. Indeed a real cubic polynomial can be expressed as the sum of two cubes of linear polynomials (Waring problem) if and only if it has two distinct complex conjugate roots or a root of multiplicity three.

2 Preliminaries

Let X be a scheme contained in a collection of double points of $\mathbf{P}^n$. We say that the type of X is $(m_1, \dots, m_{n+1})$ if X contains exactly m_i subschemes of a double point of length i, for $i = 1, \dots, n+1$. For example the type of k double points is $(0, \dots, 0, k)$. The degree of X is $\deg X = \sum i m_i$. A scheme of type $(m_1, \dots, m_{n+1})$ corresponds to a collection of linear subspaces $L_i \subseteq \mathbf{P}^n$ with $\dim L_i = i-1$ with a marked point on each L_i. Algebraic families of such schemes can be defined over any field K with the Zariski topology.

We recall now some notation and results from [BO].

Given a zero-dimensional subscheme $X \subseteq \mathbf{P}^n$, the corresponding ideal sheaf $\mathcal{I}_X$ and a linear system $\mathcal{D}$ on $\mathbf{P}^n$, the Hilbert function is defined as follows:

$$h_{\mathbf{P}^n}(X, \mathcal{D}) := \dim \mathrm{H}^0(\mathcal{D}) - \dim \mathrm{H}^0(\mathcal{I}_X \otimes \mathcal{D}).$$

If $h_{\mathbf{P}^n}(X, \mathcal{D}) = \deg X$, we say that X is $\mathcal{D}$-independent, and in the case $\mathcal{D} = \mathcal{O}_{\mathbf{P}^n}(d)$, we say d-independent.

A zero-dimensional scheme is called curvilinear if it is contained in a non singular curve. Notice that a curvilinear scheme contained in a double point has length 1 or 2.

Lemma 2.1 (Curvilinear Lemma [Ch, BO]) *Let X be a zero-dimensional scheme of finite length contained in a union of double points of $\mathbf{P}^n$ and $\mathcal{D}$ a linear system on $\mathbf{P}^n$. Then X is $\mathcal{D}$-independent if and only if every curvilinear subscheme of X is $\mathcal{D}$-independent.*

Let us denote $\mathcal{I}_X(d) = \mathcal{I}_X \otimes \mathcal{O}(d)$ and $I_X(d) = \mathrm{H}^0(\mathcal{I}_X(d))$. The expected dimension of the vector space $I_X(d)$ is $\operatorname{expdim}(I_X(d)) = \max(\binom{n+d}{n} - \deg X, 0)$.

For any scheme $X \subset \mathbf{P}^n$ and any linear subspace $L \subseteq \mathbf{P}^n$, the residual of X with respect to L is denoted by $X : L$ and it is defined by the ideal sheaf $\mathcal{I}_{X:L} = \mathcal{I}_X : \mathcal{I}_L$. We have the equality $X \cup L = (X : L) \cup L$ and the exact sequence, for any d,

$$0 \longrightarrow I_{(X:L)\cup L}(d) \longrightarrow I_X(d) \longrightarrow I_{(X\cap L),L}(d).$$

When $L = H$ is a hyperplane we obtain the well known *Castelnuovo sequence*

$$0 \longrightarrow I_{(X:H)}(d-1) \longrightarrow I_X(d) \longrightarrow I_{(X\cap H),H}(d).$$

Remark 2.2 If $Y \subseteq X \subseteq \mathbf{P}^n$ are zero-dimensional schemes, then

- if X is d-independent, then so is Y,
- if $h_{\mathbf{P}^n}(Y,d) = \binom{d+n}{n}$, then $h_{\mathbf{P}^n}(X,d) = \binom{d+n}{n}$.

It follows that if any zero-dimensional scheme $X \subseteq \mathbf{P}^n$ with $\deg X = \binom{d+n}{n}$ is d-independent, then any scheme contained in X imposes independent conditions on hypersurfaces of degree d in $\mathbf{P}^n$.

Remark 2.3 Fix $n \geq 2$ and $d \geq 3$. Assume that if a scheme X with degree $\binom{d+n}{n}$ does not impose independent conditions on hypersurfaces of degree d in $\mathbf{P}^n$, then it is of type $(m_1, \ldots, m_{n+1})$ for some given m_i. It follows that any subscheme of X is d-independent. Indeed any proper subscheme Y of X is also a subscheme of a scheme X' with degree $\binom{d+n}{n}$ and of type $(m'_1, \ldots, m'_{n+1}) \neq (m_1, \ldots, m_{n+1})$, for some m'_i and since X' is d-independent, so is Y. Moreover any scheme Z containing X impose independent conditions on hypersurfaces of degree d if it contains a scheme X'' with degree $\binom{d+n}{n}$ and of type $(m''_1, \ldots, m''_{n+1}) \neq (m_1, \ldots, m_{n+1})$ for some m''_i. Indeed since X'' imposes independent conditions on hypersurfaces of degree d, also Z does.

3 Quadratic polynomials

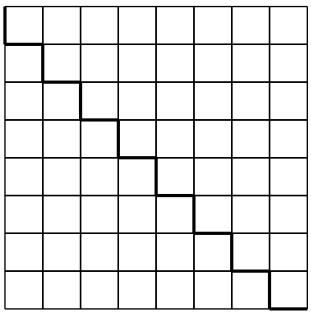
Assume that X is a scheme of type $(m_1, \ldots, m_{n+1})$. Let us fix an order on the irreducible components $\xi_1, \ldots, \xi_m$ of X (where $m = \sum m_i$) such that

$$\text{length}(\xi_1) \geq \ldots \geq \text{length}(\xi_m)$$

and for any $1 \leq i \leq m$ let us denote by l_i the length of ξ_i and by p_i the point where ξ_i is supported. Set $l_i = 0$ for $i > m$. For any $1 \leq i \leq n$ let us denote

$$\delta_X(i) = \max\{0, \sum_{j=1}^{i} l_j - \sum_{j=1}^{i}(n+2-j)\}.$$

Remark 3.1 Notice that if $\delta_X(i)$ is positive for some $1 \leq i \leq n$, then the scheme $\{\xi_1, \ldots, \xi_i\}$ does not impose independent conditions on quadrics. Indeed let be $\{e_0, \ldots, e_n\}$ a basis of V and A be the symmetric matrix defining a quadric in $\mathbf{P}(V)$ passing through the scheme $\{\xi_1, \ldots, \xi_i\}$. We may assume that $p_j = [e_{n+1-j}]$ for all $j = 1, \ldots, i$. Then, the condition $\delta_X(i) > 0$ implies that the elements in the last i columns and rows of the matrix A outlined below are all equal to 0. Hence the scheme $\{\xi_1, \ldots, \xi_i\}$ has degree $\sum_{j=1}^{i} l_j$, but imposes only $\sum_{j=1}^{i}(n+2-j) = \binom{n+2}{2} - \binom{n-i+2}{2}$ conditions on quadrics. In fact the quadrics containing $\{\xi_1, \ldots, \xi_i\}$ are exactly all the quadrics singular along the linear space spanned by $\{p_1, \ldots, p_i\}$.



The following result describes the schemes which impose independent conditions on quadrics.

Theorem 3.2 *A general zero-dimensional scheme $X \subset \mathbf{P}^n$ contained in a union of double points of type $(m_1, \ldots, m_{n+1})$ imposes independent conditions on quadrics if and only if one of the following conditions takes place:*

1. *either $\delta_X(i) = 0$ for all $1 \leq i \leq n$;*
2. *or $\deg X \geq \binom{n+2}{2} + \max\{\delta_X(i) : 1 \leq i \leq n\}$.*

Proof. First we prove that if X does impose independent conditions on quadrics, then either condition 1 or 2 holds. Assume that both conditions are false and let us prove that $I_X(2)$ has not the expected dimension $\max\{0, \binom{n+2}{2} - \deg(X)\}$. Indeed choose $i \in \{1, \ldots, n\}$ such that $\delta_X(i) > 0$ and $\deg X < \binom{n+2}{2} + \delta_X(i)$ and consider the family $\mathcal{C}$ of quadratic cones with vertex containing the linear space $\mathbf{P}^{i-1}$ spanned by $p_1, \ldots, p_i$. Of course we have

$$\dim I_X(2) \geq \dim(\mathcal{C}) - (\deg(X) - \sum_{j=1}^{i} l_j) = \binom{n-k+2}{2} - \deg(X) + \sum_{j=1}^{i} l_j = c.$$

Now we compute

$$\dim I_X(2) - \mathrm{expdim} I_X(2) \geq \min\{c, \binom{n-k+2}{2} + \sum_{j=1}^{i} l_j - \binom{n+2}{2}\} = \min\{c, \delta_X(i)\}$$

By assumption $\delta_X(i) > 0$ and

$$c > \binom{n-k+2}{2} - \binom{n+2}{2} - \delta_X(i) + \sum_{j=1}^{i} l_j = \sum_{j=1}^{i} l_j - \sum_{j=1}^{i} (n+2-j) - \delta_X(i) = 0$$

Hence the dimension of $I_X(2)$ is higher than the expected dimension.

Now we want to prove that if either condition 1 or condition 2 holds, then X imposes independent conditions on quadrics. We work by induction on $n \geq 2$. If $n = 2$ it is easy to check directly our claim.

Consider a scheme X in $\mathbf{P}^n$ which satisfies condition 1 and fix a hyperplane $H \subset \mathbf{P}^n$. We specialize all the components of X on H in such a way that the residual of each of the components $\xi_1, \ldots, \xi_{n+1}$ is 1 (if they are not zero) and the residual of the remaining components is zero. Then we get the Castelnuovo sequence

$$0 \to I_{X:H}(1) \to I_X(2) \to I_{X \cap H}(2)$$

where $X : H$ is the residual given by at most $n+1$ simple points and $X \cap H$ is the trace in H. Hence we conclude by induction once we have proved that the trace $X \cap H$ satisfies condition 1 or 2.

In fact we prove now that if $X \cap H$ does not satisfy condition 1, then it satisfies 2. Assume that for X in $\mathbf{P}^n$ we have $\delta_X(i) = 0$ for all $1 \leq i \leq n$, while for $X \cap H$ in H we have $\delta_{X\cap H}(i) > 0$ for some $1 \leq i \leq n-1$. Choose the index $1 \leq k+h \leq n-1$ such that $\delta_{X\cap H}(i)$ is maximum. This is possible only if there is some index $1 \leq k \leq k+h \leq n-1$ such that $l_k = l_{k+1} = \ldots = l_n = l_{n+1} = \ldots = l_{n+1+h} = l$. Since $\delta_X(k+h) = 0$ and $\delta_{X\cap H}(k+h) = 0$ we obtain

$$\sum_{i=1}^{k+h}(n+2-i) - h < \sum_{i=1}^{k+h} l_i \leq \sum_{i=1}^{k+h}(n+2-i),$$

from which it follows that

$$l \geq (n+2-k-h). \tag{2}$$

Now in order to prove that $X \cap H$ satisfies 2 we need to show that

$$\deg X \cap H \geq \binom{n+1}{2} + \delta_{X\cap H}(k+h).$$

Notice that

$$\deg X \cap H \geq \deg X - (n+1) \geq \sum_{i=1}^{n+1+h} l_i - (n+1),$$

hence if we prove the following inequality we are done:

$$\sum_{i=1}^{n+1+h} l_i - (n+1) \geq \binom{n+1}{2} + \delta_{X\cap H}(k+h)$$

i.e.

$$\sum_{i=1}^{k-1} l_i + \sum_{i=k}^{n+1+h} l - (n+1) \geq \binom{n+1}{2} + \sum_{i=1}^{k-1}(l_i - 1) + \sum_{i=k}^{k+h} l - \sum_{i=1}^{k+h}(n+1-i)$$

which reduces to

$$(n+1-k)l - (n+1) \geq \binom{n+1}{2} - (k-1) - (k+h)(n+1) + \binom{k+h+1}{2}.$$

By using inequality (2) it is enough to prove, for any $n \geq 2$, any $1 \leq k \leq n-1$ and any $0 \leq h \leq n-1-k$, that

$$(n+1-k)(n+2-k-h) \geq \binom{n+1}{2} + (n-k) - (k+h)(n+1) + \binom{k+h+1}{2} \tag{3}$$

and we prove this inequality by induction on h.

First fix n, k and choose $h = n-1-k$. In this case (3) becomes

$$3(n+1-k) \geq \binom{n+1}{2} + (n-k) - (n-1)(n+1) + \binom{n}{2}$$

which is true. Now if we assume that (3) is verified for $h = h' \leq n-1-k$, it is easy to check it for $h = h'-1$, thus completing the proof of (3).

It remains to prove that if X satisfies condition 2, then the system of quadrics $|\mathcal{I}_X(2)|$ containing X is empty.

If $\delta_X(i) = 0\ \forall i$ then we are in the previous case. We may assume that there exists i such that $\delta_X(i) > 0$. If the sequence $\{\delta_X(i)\}$ is nondecreasing then in particular $\delta_X(n) > 0$. This implies easily that the quadrics containing the first n components $\{\xi_1, \ldots, \xi_n\}$ are singular along the hyperplane $H =< p_1, \ldots, p_n >$, so the only existing quadric is the double hyperplane H^2. By assumption $\deg X > \left[\binom{n+2}{2} - 1\right] + \delta_X(n)$, hence there is an extra condition and $|\mathcal{I}_X(2)| = \emptyset$ as we wanted.

We may assume that there exists $i < n$ such that $\delta_X(i+1) < \delta_X(i)$ and we pick the first such i. In particular it follows

$$l_{i+1} < n+1-i \tag{4}$$

As above, all the quadrics containing $X_0 = \{\xi_1, \ldots, \xi_i\}$ are singular along the linear space $L_0 =< p_1, \ldots, p_i >$. Let $X_1 = X \setminus X_0$. By definition $\deg X_0 = \binom{n+2}{2} - \binom{n+2-i}{2} + \delta_X(i)$.

Let π be the projection from L_0 on a linear space $L_1 \simeq \mathbf{P}^{n-i}$. By (4) we have $\deg X_1 = \deg \pi(X_1)$. Hence there is a bijective correspondence between $|\mathcal{I}_X(2)|$ and $|\mathcal{I}_{\pi(X_1)}(2)| \subseteq |\mathcal{O}_{L_1}(2)|$.

Note that

$$\deg \pi(X_1) - \binom{n+2-i}{2} = \deg X - \binom{n+2}{2} - \delta_X(i) \geq \max_h\{\delta_X(h)\} - \delta_X(i) \geq 0$$

hence if $\delta_{\pi(X_1)}(h) = 0$ for $h = 1, \ldots, n-i$ we conclude again by the first case. If there exists j such that $\delta_{\pi(X_1)}(j) > 0$, notice that in such a case we have

$$\delta_{\pi(X_1)}(j) = \delta_X(j+i) - \delta_X(i),$$

hence

$$\max_p\{\delta_{\pi(X_1)}(p)\} = \max_h \delta_{\pi(X_1)}(h) - \delta_X(i).$$

We get that

$$\deg \pi(X_1) - \binom{n+2-i}{2} \geq \max_p\{\delta_{\pi(X_1)}(p)\}$$

This means that $\pi(X_1)$ satisfies the assumption 2 on L_1 and then by (complete) induction on n we get that $|\mathcal{I}_{\pi(X_1)}(2)| = \emptyset$ as we wanted. □

A straightforward consequence of the previous theorem is the following corollary.

Corollary 3.3 *A general zero-dimensional scheme $X \subset \mathbf{P}^n$ contained in a union of double points with $\deg X = \binom{n+2}{2}$ imposes independent conditions on quadrics if and only if $\delta_X(i) = 0$ for all $1 \leq i \leq n$.*

Theorem 3.2 provides a classification of all the types of general subschemes X of a collection of double points of $\mathbf{P}^n$ which do not impose independent conditions on quadrics. For example in $\mathbf{P}^2$, the only case is X given by two double points. In $\mathbf{P}^3$ and in $\mathbf{P}^4$ we have the following lists of subschemes which do not impose independent conditions on quadrics.

X	$\deg X$	$\max\{\delta_X(i)\}$	$(m_1, \ldots, m_4)$	$\dim I_X(2)$
4,4,4	12	3	$(0,0,0,3)$	1
4,4,3	11	2	$(0,0,1,2)$	1
4,4,2	10	1	$(0,1,0,2)$	1
4,4,1,1	10	1	$(2,0,0,2)$	1
4,4,1	9	1	$(1,0,0,2)$	2
4,4	8	1	$(0,0,0,2)$	3
4,3,3	10	1	$(0,0,2,1)$	1

Table 1: List of exceptions in $\mathbf{P}^3$

X	$\deg X$	$\max\{\delta_X(i)\}$	$(m_1, \ldots, m_5)$	$\dim I_X(2)$
5,5,5,5	20	6	$(0,0,0,0,4)$	1
5,5,5,4	19	5	$(0,0,0,1,3)$	1
5,5,5,3	18	4	$(0,0,1,0,3)$	1
5,5,5,2	17	3	$(0,1,0,0,3)$	1
5,5,5,1,1	17	3	$(2,0,0,0,3)$	1
5,5,5,1	16	3	$(1,0,0,0,3)$	2
5,5,5	15	3	$(0,0,0,0,3)$	3
5,5,4,4	18	4	$(0,0,0,2,2)$	1
5,5,4,3	17	3	$(0,0,1,1,2)$	1
5,5,4,2	16	2	$(0,1,0,1,2)$	1
5,5,4,1,1	16	2	$(2,0,0,1,2)$	1
5,5,4,1	15	2	$(1,0,0,1,2)$	2
5,5,4	14	2	$(0,0,0,1,2)$	3
5,5,3,3	16	2	$(0,0,2,0,2)$	1
5,5,3,2	15	1	$(0,1,1,0,2)$	1
5,5,3,1,1	15	1	$(2,0,1,0,2)$	1
5,5,3,1	14	1	$(1,0,1,0,2)$	2
5,5,3	13	1	$(0,0,1,0,2)$	3
5,5,2,2,1	15	1	$(1,2,0,0,2)$	1
5,5,2,2	14	1	$(0,2,0,0,2)$	2
5,5,2,1,1,1	15	1	$(3,1,0,0,2)$	1
5,5,2,1,1	14	1	$(2,1,0,0,2)$	2
5,5,2,1	13	1	$(1,1,0,0,2)$	3
5,5,2	12	1	$(0,1,0,0,2)$	4
5,5,1,1,1,1,1	15	1	$(5,0,0,0,2)$	1
5,5,1,1,1,1	14	1	$(4,0,0,0,2)$	2
5,5,1,1,1	13	1	$(3,0,0,0,2)$	3
5,5,1,1	12	1	$(2,0,0,0,2)$	4
5,5,1	11	1	$(1,0,0,0,2)$	5
5,5	10	1	$(0,0,0,0,2)$	6
5,4,4,2	15	1	$(0,1,0,2,1)$	1
5,4,4,1,1	15	1	$(2,0,0,2,1)$	1
5,4,4,1	14	1	$(1,0,0,2,1)$	2
5,4,4	13	1	$(0,0,0,2,1)$	3
4,4,4,4	16	2	$(0,0,0,4,0)$	1
4,4,4,3	15	1	$(0,0,1,3,0)$	1

Table 2: List of exceptions in $\mathbf{P}^4$

4 Cubic polynomials

In this section we generalize the approach of [BO, Section 3] to our setting and we prove the following result.

Theorem 4.1 *A general zero-dimensional scheme $X \subset \mathbf{P}^n$ contained in a union of double points imposes independent conditions on cubics with the only exception of $n = 4$ and X given by 7 double points.*

First we give the proof of the previous theorem in cases $n = 2, 3, 4$.

Lemma 4.2 *Let be $n = 2, 3$ or 4. Then a general zero-dimensional scheme $X \subset \mathbf{P}^n$ contained in a union of double points imposes independent conditions on cubics with the only exception of $n = 4$ and X given by 7 double points.*

Proof. By Remark 2.2 it is enough to prove the statement for X with degree $\binom{n+3}{3}$.

Let $n = 2$ and X a subscheme of a collection of double points with $\deg X = 10$. Fix a line H in $\mathbf{P}^2$ and consider the Castelnuovo exact sequence

$$0 \to I_{X:H}(2) \to I_X(3) \to I_{X\cap H}(3).$$

It is easy to prove that it is always possible to specialize some components of X on H so that $\deg(X \cap H) = 4$ and that the residual $X : H$ does not contain two double points. The last condition ensures that $\delta_{X:H}(i) = 0$ for $i = 1, 2$. Hence we conclude by Corollary 3.3.

In the case $n = 3$, the scheme X has degree 20. Since there are no cubic surfaces with five singular points we can assume that X contains at most three double points. Indeed if X contains 4 double points we can degenerate it to a collection of 5 double points. We fix a plane H in $\mathbf{P}^3$ and we want to specialize some components of X on H so that $\deg(X \cap H) = 10$ and that the residual $X : H$ imposes independent conditions on quadrics. By looking at Table 1, since $\deg(X : H) = 10$, it is enough to require that $X : H$ is not of the form $(0, 1, 0, 2), (2, 0, 0, 2)$ or $(0, 0, 2, 1)$. It is easy to check that this is always possible: indeed specialize on H the components of X starting from the ones with higher length and keeping the residual as minimal as possible until the degree of the trace is 9 or 10. If the degree of the trace is 9 and there is in X a component with length 1 or 2 we can obviously complete the specialization. The only special case is given by X of type $(0, 0, 3, 2)$ and in this case we specialize on H the two double points and two components of length 3 so that each of them has residual 1.

If $n = 4$ the case of 7 double points is exceptional. Assume that X has degree 35 and contains at most 6 double points. We fix a hyperplane H of $\mathbf{P}^4$ and we want to specialize some components of X on H so that $\deg(X \cap H) = 20$ and that the residual $X : H$ imposes independent conditions on quadrics. By looking at Table 2, it is enough to require that $X : H$ does not contain two double points, does not contain one double point and two components of length 4 and it is not of the form $(0, 0, 1, 3, 0)$. It is possible to satisfy this conditions by specializing the components of X in the following way: we specialize the components of X on H starting from the ones with higher length and keeping the residual as minimal as possible until the degree of the trace is maximal and does not exceed 20. Then we add some components allowing them to have residual 1 in order to reach the degree 20. It is possible to check that this construction works, except for the case $(0, 0, 5, 0, 4)$ where we have to specialize on H all the double points and 2 of the components with length 3 so that both have residual 1. It is easy also to check

that following the construction above the residual has always the desired form, except for X of the form $(0,0,1,8,0)$, where the above rule gives a residual of type $(0,0,1,3,0)$. In this case we make a specialization ad hoc: for example we can put on H six components of length 4 and the unique component of length 3 in such a way that all them have residual 1 and we obtain a residual of type $(7,0,0,2,0)$ which is admissible.

Now we have to check the schemes either contained in 7 double points or containing 7 double points. But this follows immediately by Remark 2.3. □

Let $L \subset \mathbf{P}^n$ be a codimension three linear subspace ($n \geq 3$). In order to prove Theorem 4.1 by induction we need to consider linear systems of hypersurfaces of degree d (for $d \leq 3$) through the subspace L. Note that L imposes $n-2$ conditions on hyperplanes, $\binom{n-1}{2}$ conditions on quadrics and $\binom{n}{3}$ conditions on cubics.

The proof of Theorem 4.1 relies on the following description.

Proposition 4.3 *Assume $n \geq 3$ and let X be a zero-dimensional scheme of $\mathbf{P}^n$ contained in the union of double points. Let X_L be the part of X supported on L and let $X_O = X \setminus X_L$. Assume that X_O is general and X_L is general among the schemes supported on L. Then the following assertions hold.*

(i) If $\deg X : L \leq 3$ then $X \cup L$ imposes $\deg X : L + (n-2)$ independent conditions on hyperplanes.

(ii) If $\deg X : L \leq 3n$ and X_O does not contain two double points, then $X \cup L$ imposes $\deg X : L + \binom{n-1}{2}$ independent conditions on quadrics.

(iii) If $\deg X : L = \frac{3}{2}n^2 + \frac{3}{2}n + 1$, X_O does not contain a subscheme with degree greater than or equal to $(n+1)(n+2)-1$ given by $n+2$ components and $\deg X_L : L \leq \frac{n(n-1)}{2} + n$, (so that $\deg X_O \geq (n+1)^2 - n$), then $X \cup L$ imposes $\deg X : L + \binom{n}{3}$ independent conditions on cubics.

Proof. Part (i) is trivial. In order to prove (ii) we may assume that the degree of $X : L$ is $3n$. Let A be the symmetric matrix defining the quadric. Observe that the condition that the quadric contains L implies in a convenient system of coordinates that the submatrix of A defined by the first $n-2$ rows and by the first $n-2$ columns is all zero. Thus if $X : L$ has only three irreducible components, then the statement follows easily by looking at the last three columns of A.

In the general case we work by induction on n. Let $H \subset \mathbf{P}^n$ be a hyperplane transverse to L. We rely on the sequence

$$0 \longrightarrow I_{(X:H)\cup L}(1) \longrightarrow I_{X\cup L}(2) \longrightarrow I_{(X\cup L)\cap H}(2)$$

and we specialize the first three irreducible components with higher length of $X : L$ to H in such a way that each of them has a residual of length one, (this is always possible except for the trivial case of many simple points). All the other irreducible components of $X : L$ have length smaller than or equal to $n-1$, thus they may be specialized inside H and we conclude by (i). The starting point of the induction is $n = 3$, which is follows from Theorem 3.2.

Now we prove statement (iii). If $n = 3$ the statement is trivial, by Lemma 4.2. For $n \geq 4$ we proceed by induction. Given a hyperplane H we want to specialize some connected components of $X : L$ to H in such a way that the residual of $X : L$ with respect to H has length $3n$ and does not contain two double points and such that the trace of $X : L$ on H satisfies the assumptions of (iii) . This allows us to conclude the proof by induction on n, by relying on (ii) and on the sequence

$$0 \longrightarrow I_{(X:H)\cup L}(2) \longrightarrow I_{X\cup L}(3) \longrightarrow I_{(X\cup L)\cap H}(3).$$

Now we show that such a specialization is always possible. When we specialize the components of X, we keep on L the ones supported on L and outside L the ones supported outside L. Hence $(X \cap H)_L = X_L \cap H$ and $(X \cap H)_O = (X_O \cap H)$. Now we consider the following cases:

1. If $\deg X_L : L \leq \frac{n(n-1)}{2} - (n-1)$, we specialize X_L on H so that any component of $X_L : L$ is contained in H.
2. If $\frac{n(n-1)}{2} - (n-1) + 1 \leq \deg X_L : L \leq \frac{n(n-1)}{2} + n - 1$, and $X_L : L$ has at least $n-1$ components, we specialize X_L on H so that exactly $n-1$ component of $X_L : L$ have residual 1 with respect to H and all the remaining have residual 0.

   If $X_L : L$ has less than $n-1$ components, then we proceed as follows. If there is a component of $X_L : L$ with length 3, we keep it outside H and we specialize all the remaining components on H so that exactly $n-4$ of them have residual 1 with respect to H. Indeed since $\deg X_L : L \geq \frac{(n-1)(n-2)}{2} + 1$, there exist at least other $n-4$ components in $X_L : L$. If there are no component of $X_L : L$ with length 3, since $X_L : L$ has less than $n-1$ components, it follows that $n = 4$. In this case, if there is a component of length 2, we keep it outside H and we specialize all the remaining components on H so that exactly one of them has residual 1 with respect to H. Indeed since $\deg X_L : L \geq 4$ there is at least another component. If there are no components of length 2, we have at least 3 component of length 1 and we can conclude.
3. If $\deg X_L : L = \frac{n(n-1)}{2} + n$ and $n \geq 5$, then $X_L : L$ has at least n components and we specialize X_L on H so that exactly n component of $X_L : L$ have residual 1 and all the remaining have residual 0. If $n = 4$ we can always make a specialization such that the residual with respect H of $X_L : L$ has degree 4.

In all the previous cases the trace $X \cap H$ satisfies the condition

$$\deg(X_L \cap H) : L \leq \frac{(n-1)(n-2)}{2} + (n-1).$$

Now we want to specialize X_O in such a way that the residual $X_O : H$ has degree $3n$ in the first case, $2n+1$ in the second one, $2n$ in the third one. More precisely we proceed as follows.

In case 1 we have by assumption that $\deg X_O \geq (n+1)^2 + n$, hence X_O has at least $n+3$ components. If X_O contains double points we keep one of them outside H and we specialize all the remaining double points on H making them having residual 1. Then we keep some of the remaining components outside H, in such a way that $\deg X_O : H = 3n$. If X_O does not contain double points we keep outside H some of the components with maximal degree and we specialize all the remaining components in H in such a way that some of them have residual one and that $\deg X_O : H = 3n$. It is easy to check that this is always possible.

In case 2 we want to specialize X_O in such a way that $X_O : H$ has degree $2n+1$. We proceed as follows. If X_O contains double points, we keep one of them outside H and we specialize all the remaining components on H, making exactly n of them having residual 1. If there are no double points in X_O we proceed as follows: we keep outside H one irreducible component with maximal length $1 \leq k \leq n$ and we specialize all the remaining components on H, making exactly

$2n+1-k$ of them having residual 1. This is possible since the inequality

$$\left\lceil \frac{\deg X_O - k}{k} \right\rceil \geq 2n+1-k \tag{5}$$

holds for any $1 \leq k \leq n$. In fact we prove that (5) holds if $\deg X_O \geq n^2+n+1$. Indeed it is enough to prove that $\left\lceil \frac{n^2+n+1-k}{k} \right\rceil \geq 2n+1-k$ holds for any $1 \leq k \leq n$. Let be $k = n-h$ for some $0 \leq h \leq n-1$. Since

$$n^2+n+1 = (n-h)^2 + (2h+1)(n-h) + h^2 + h + 1$$

it is clear that

$$\left\lceil \frac{n^2+n+1-k}{k} \right\rceil = (n-h)+(2h+1)-1+\left\lceil \frac{h^2+h+1}{k} \right\rceil \geq n+h+1 = 2n+1-k$$

and (5) is proved.

In case 3 we want to specialize X_O in such a way that $X_O : H$ has degree $2n$. Notice that in this case we have $\deg X_O = (n+1)^2 - n$. Hence X_O does not contain more than n double points, and contains at least $n+1$ components. If X_O contains double points, we keep one of them outside H and we specialize all the remaining components on H, making exactly $n-1$ of them having residual 1. If X_O does not contain double points, we keep outside H one irreducible component with maximal length $1 \leq k \leq n$ and we specialize all the remaining components on H, making exactly $2n-k$ of them having residual 1. Since $\deg X_O = n^2+n+1$, we have proved in the previous case that the inequality (5) holds. Then we can apply induction as we claimed. □

Proof of Theorem 4.1. We prove the claim by induction on n by step 3. By Lemma 4.2 we know that the theorem holds for $n = 2, 3, 4$.

Let be X a general scheme contained in a collection of double points and with $\deg X = \binom{n+3}{3}$. Fix a codimension three linear subspace $L \subset \mathbf{P}^n$ and consider the exact sequence

$$0 \longrightarrow I_{(X:L)\cup L}(3) \longrightarrow I_X(3) \longrightarrow I_{(X\cap L),L}(3). \tag{6}$$

We want to specialize on L some components of X and apply part (iii) of Proposition 4.3.

Let us fix an order on the irreducible components $\xi_1, \ldots, \xi_m$ of X such that $\text{length}(\xi_i) \geq \text{length}(\xi_{i+1})$. Then we specialize on L the components ξ_i starting from ξ_1 in such a way that the residual with respect to L of each component is minimum and the trace on L has degree smaller than or equal to $\binom{n}{3}$. Let ξ_k be the last component we have specialized. If

$$\deg(\{\xi_1 \cup \ldots \cup \xi_k\} \cap L) < \binom{n}{3},$$

we specialize on L also the component ξ_{k+1}, keeping its residual smaller as possible. Let us denote by X_L the components of X supported on L and $X_O = X \setminus X_L$. Now we want to prove that the following two conditions are fulfilled:

1. $\deg(X_L : L) \leq \frac{n(n-1)}{2} + n$
2. X_O does not contain a subscheme supported in $n+2$ points and of degree greater than n^2+3n.

In order to prove 1, assume that d (e, f respectively) is the number of double points (of components of length n, $n-1$ respectively) contained in the set $\{\xi_1 \cup \ldots \cup \xi_k\}$. Clearly we have

$$\deg(X_L : L) = \deg(\{\xi_1 \cup \ldots \cup \xi_k\} : L) + \deg(\xi_{k+1} : L) \leq 3d + 2e + f + n$$

and it is enough to show that $3d+2e+f \leq \frac{n(n-1)}{2}$. If by contradiction we assume $3d + 2e + f > \frac{n(n-1)}{2}$, then we have

$$d + e + f > \frac{n(n-1)}{6} \tag{7}$$

On the other hand we have the inequalities

$$(d + e + f)(n - 2) \leq \deg(X_L \cap L) \leq \binom{n}{3}$$

from which we get $d + e + f \leq \frac{n(n-1)}{6}$ which contradicts (7).

In order to prove 2 we need to check that X_O contains neither $n + 2$ double points, nor $n + 1$ double points and a component of length n. Since the maximal number of double points in X is $\lceil \frac{(n+3)(n+2)}{6} \rceil$ and we have to specialize on L at least $\frac{n(n-1)}{6}$ of them, it is impossible that in X_O remain more than $n + 1$ double points. In the case that we have exactly $n+1$ double points in X_O, then the union of the remaining components of X_O has degree $\binom{n+3}{3} - (n+1)\frac{n(n-1)}{6} - (n+1)^2 = 0$ This implies that we can always specialize X so that the residual $X : L$ satisfies the assumptions of part (iii) of Proposition 4.3.

In order to complete the proof we need to study the case $n = 7$. If X is given by 15 double points, we conclude by Alexander-Hirschowitz theorem. If X is a subscheme of $\mathbf{P}^7$ with degree 120 and with at most 14 double points, we prove that X imposes independent conditions on cubics by relying on the exact sequence (6). Indeed it is easy to show that we can always specialize some components on L so that the trace has degree 35 and is not given by 7 double points, and so that the residual satisfies the assumptions of part (iii) of Proposition 4.3.

Finally notice that by Remark 2.3 all the schemes either contained in 15 double points or containing 15 double points impose independent conditions on cubics. □

5 Induction

In order to prove Theorem 1.1 we will work by induction on the dimension and the degree. In the following lemmas we describe case by case the initial and special instances, while in Theorem 5.6 below we present the general inductive procedure, which involves the differential Horace method.

Lemma 5.1 *A general zero-dimensional scheme $X \subset \mathbf{P}^2$ contained in a union of double points imposes independent conditions on $\mathcal{O}_{\mathbf{P}^2}(d)$ for any $d \geq 4$, with the only exception of $d = 4$ and X given by the collection of 5 double points.*

Proof. Assume that X is a general subscheme of a union of double points with $\deg(X) = \binom{d+2}{2}$. If X is a collection of double points the statement follows from the Alexander-Hirschowitz theorem on $\mathbf{P}^2$ (for an easy proof see for example [BO, Theorem 2.4]).

If X is not a collection of double points, fix a hyperplane $\mathbf{P}^1 \subset \mathbf{P}^2$. Since X contains at least a component of length 1 or 2, it is clearly always possible to find a specialization of X such that the trace has degree exactly $d+1$. Then we conclude by induction from the *Castelnuovo sequence*

$$0 \to I_{X:\mathbf{P}^1}(d-1) \to I_X(d) \to I_{X\cap\mathbf{P}^1}(d).$$

Notice that any subscheme of 5 double points and any scheme containing 5 double points impose independent conditions on quartics, by Remark 2.3. □

We give now an easy technical lemma that we need in the following.

Lemma 5.2 *Assume that X is a general zero-dimensional scheme contained in a union of double points of $\mathbf{P}^n$, which contains at least $n-1$ components of length less than or equal to n. Then if $\deg(X) = \binom{n+d}{n}$ it is possible to specialize some components of X on a fixed hyperplane $\mathbf{P}^{n-1}$ in such a way that $\deg(X \cap \mathbf{P}^{n-1}) = \binom{n-1+d}{n-1}$.*

Proof. By assumption there exist at least $n-1$ components $\{\eta_1, \ldots, \eta_{n-1}\}$ with $\text{length}(\eta_i) \leq n$. Specialize $\eta_1, \ldots, \eta_{n-1}$ on the hyperplane $\mathbf{P}^{n-1}$ in such a way that the residual of each component is zero. Then specialize other components so that

$$\delta = \binom{n-1+d}{n-1} - \deg(X \cap \mathbf{P}^{n-1}) \geq 0$$

is minimal. If $\delta = 0$ the claim is proved, so assume $\delta \geq 1$. Obviously we have $\delta < k-1 \leq n$, where k is the minimal length of the components of X which lie outside $\mathbf{P}^{n-1}$. Let ζ be a component with length k. Now we make the first components $\eta_1, \ldots, \eta_{k-1-\delta}$ having residual 1 with respect to $\mathbf{P}^{n-1}$ and we specialize ζ on $\mathbf{P}^{n-1}$ with residual 1. Notice that this is possible since $0 < k-1-\delta \leq n-1$. □

Lemma 5.3 *Fix $3 \leq n \leq 4$. A general zero-dimensional scheme $X \subset \mathbf{P}^n$ contained in a union of double points imposes independent conditions on $\mathcal{O}_{\mathbf{P}^n}(4)$, with the following exceptions:*

- *$n = 3$ and either X is the union of 9 double points, or X is the union of 8 double points and a component of length 3;*
- *$n = 4$ and X is the union of 14 double points.*

Proof. If X is a collection of double points, the statement holds by the Alexander-Hirschowitz theorem. Assume that X is a scheme with degree $\binom{n+4}{4}$ which is not a union of double points. Let us denote by D the number of double points in X and by C the number of the components with length less than or equal to n.

If $n = 3$ and $C = 1$, then $D = 8$ and X is the exceptional case of the statement. If $n = 3$ and $C = 2$, then $D = 8$ and the two components η_1 and η_2 with length less than or equal to 3 have necessarily length 1 and 2. In this case we specialize X on $\mathbf{P}^2$ in such a way that the trace is given by the union of η_1, η_2 and 4 double points. Hence we conclude by the Castelnuovo sequence and by induction. If $C \geq 3$, then we denote by η the component of X with minimal length. We specialize η on $\mathbf{P}^2$ in such a way that its residual is 1 if $\text{length}(\eta) \geq 2$, and 0 if η is a simple point. Then we apply the construction of Lemma 5.2 on $X \setminus \eta$ (which has at least two components with length less than or equal to 3) and we obtain a trace different from 5 double points. Hence we conclude by the Castelnuovo sequence and by induction.

If $n = 4$ and $C = 2$, then X is given either by the union of 13 double points, a component of length 3 and one of length 2, or by the union of 13 double points, a component of length 4 and a simple point. In the first case we specialize X obtaining a trace given by 8 double points, a component of length 2 and a simple point. Then we conclude by induction as before. In the second case we cannot use the Castelnuovo sequence since we would obtain an exceptional case. In order to conclude we prove that a general union of 13 double points and a component of length 4 imposes independent conditions on quartics. Indeed we know that there exists a unique quartic hypersurface through 14 double points supported at $p_1, \ldots, p_{14}$. This implies that for any $i = 1, \ldots, 14$ there is a unique line r_i through p_i such that $r_1, \ldots, r_{14}$ are contained in a hyperplane. Then we consider the scheme Y given by the union of 13 double points supported at $\{p_1, \ldots, p_{13}\}$ and the component of length 4 corresponding to a linear space of dimension 3 which does not contain r_{14}. It is clear that the scheme Y imposes independent conditions on quartics, then also the scheme given by the union of Y and a general simple point does the same.

Assume now that $n = 4$ and $C = 3$. If $D = 13$, then we can degenerate X to one of the previous cases where the components with length less than or equal to 4 are two. If $D = 12$, then the remaining three components have length either $3, 3, 4$ or $2, 4, 4$. In these cases we can obtain as a trace 7 double points and three components of length either $2, 2, 3$, or $1, 3, 3$, and we conclude by the Castelnuovo sequence.

If $n = 4$ and $C \geq 4$, we denote by η the component of X with minimal length. If $\text{length}(\eta) = 1$ we can degenerate X to a scheme X' where the components with length less than or equal to 4 are one less and we apply the argument to X'. If $2 \leq \text{length}(\eta) \leq 3$, then we specialize η on $\mathbf{P}^3$ in such a way that the residual of η is 1. Then we apply the construction of Lemma 5.2 on $X \setminus \eta$ (which has at least three components with length less than or equal to 3) and we obtain a trace different from 8 double points and a component of length 3. Moreover with this construction we always avoid a residual given by 7 double points. Hence we conclude by the Castelnuovo sequence. If $\text{length}(\eta) = 4$, we have only the following possibilities: 5 components of length 4 and 10 double points, 10 components of length 4 and 6 double points, 15 components of length 4 and 2 double points. In the first two cases we can obtain trace on $\mathbf{P}^3$ given by 5 components of length 3 and 5 double points, while in the third case we can obtain a trace equal to 9 components of length 3 and 2 double points. Then we conclude by the Castelnuovo sequence. □

Lemma 5.4 *Fix $5 \leq n \leq 9$. A general zero-dimensional scheme $X \subset \mathbf{P}^n$ contained in a union of double points imposes independent conditions on $\mathcal{O}_{\mathbf{P}^n}(4)$.*

Proof. If X is a collection of double points, the statement holds by the Alexander-Hirschowitz theorem. Assume that X is a scheme with degree $\binom{n+4}{4}$ which is not a union of double points. Let us denote by D the number of double points in X and by C the number of the components with length less than or equal to n.

If $n \in \{5, 6, 8\}$ and $C = 2$, then we conclude by degenerating X to a union of double points.

If $n = 5$ and $C = 3$, then we get either $D = 20$ or $D = 19$. In the first case we conclude degenerating X to the union of 21 double points. In the second case the remaining three components have length $2, 5, 5$, or $3, 4, 5$, or $4, 4, 4$. Then

we can obtain a trace equal to 12 double points and three components of length respectively $2, 4, 4$ in the first case, or $3, 3, 4$ in the second and third cases. Then we conclude by induction.

If $n = 5$ and $C = 4$, then we have $D \in \{20, 19, 18\}$. In the first case we can degenerate X to a union of 21 double points. If X can be degenerate to a scheme which contains only three components with length less than or equal to 5, we conclude by using the previous results. Then we have to consider only the two cases where X is given by 18 double points and four components of length either $3, 5, 5, 5$, or $4, 4, 5, 5$. In these cases we can obtain a trace equal to 12 double points and three components of length respectively $2, 4, 4$ in the first case, and $3, 3, 4$ in the second case. Hence we conclude by induction.

If $n = 5$ and $C \geq 5$, we denote by η the component with minimal length. Then we specialize η on $\mathbf{P}^4$ in such a way that the residual of η is 1 if η if length$(\eta) \geq 2$, and 0 if η is a simple point. Then we apply the construction of Lemma 5.2 on $X \setminus \eta$ (which has at least four components with length less than or equal to 5) and we obtain a trace different from 14 double points. Hence we conclude by the Castelnuovo sequence and by induction.

If $n = 6$ and $D \geq 21$, we specialize 21 double points on $\mathbf{P}^5$ and we conclude by the Castelnuovo sequence. If $D < 21$, then we have $C \geq 5$ and we can apply Lemma 5.2, concluding by the Castelnuovo sequence.

If $n = 7$ and $D \geq 30$, we specialize 30 double points on $\mathbf{P}^6$ and we conclude by the Castelnuovo sequence. If $D < 30$, then we have $C \geq 6$ and we can apply Lemma 5.2.

If $n = 8$ and $C = 3$, then either $D = 58$ and X can be degenerated to the union of 59 double points, or $D = 57$. In this case the remaining three components can have length $5, 5, 8$, or $5, 6, 7$, or $6, 6, 6$. In all these case we can obtain a trace on $\mathbf{P}^7$ given by 40 double points and two components of total degree 10.

If $n = 8$ and $C = 4$ and X can be degenerated to a scheme with less than 4 components with length less than or equal to 8, then we conclude. Then we have only to consider the case where $D = 56$ and the remaining four components of X have length $3, 8, 8, 8$, or $4, 7, 8, 8$, or $5, 6, 8, 8$, or $5, 7, 7, 8$, or $6, 6, 7, 8$, or $6, 7, 7, 7$. In all these cases we obtain a trace on $\mathbf{P}^7$ given by 40 double points and two components of total degree 10, with the exception of the last case, where we can obtain a trace given by 39 double points and three components of total degree 18.

If $n = 8$ and $C = 5$ and X can be degenerated to a scheme with less than 5 components with length less than or equal to 8, then we conclude. Hence we have only to consider the cases $D = 56$ or $D = 55$. Listing all the possible lengths of the remaining five components we easily notice that we can always obtain a trace on $\mathbf{P}^7$ given either by 40 double points and two components of total degree 10, or by 39 double points and three components of total degree 18.

If $n = 8$ and $C = 6$ and X can be degenerated to a scheme with less than 6 components with length less than or equal to 8, then we conclude. Hence we have only to consider the cases $D = 55$ or $D = 54$. Listing all the possible lengths of the remaining six components, we easily notice, as before, that we can always obtain a trace on $\mathbf{P}^7$ given either by 40 double points and two components of total degree 10, or by 39 double points and three components of total degree 18.

If $n = 8$ and $C \geq 7$, we apply Lemma 5.2 and we conclude by the Castelnuovo sequence.

If $n = 9$ and $D \geq 59$, we specialize 59 double points on $\mathbf{P}^8$ and we conclude by the Castelnuovo sequence. If $D < 59$, then we get $C \geq 8$ and we conclude by applying Lemma 5.2 and by the Castelnuovo sequence. □

Lemma 5.5 *Fix $3 \leq n \leq 4$ and $5 \leq d \leq 6$. A general zero-dimensional scheme $X \subset \mathbf{P}^n$ contained in a union of double points imposes independent conditions on $\mathcal{O}_{\mathbf{P}^n}(d)$.*

Proof. If X is a collection of double points, the statement holds by the Alexander-Hirschowitz theorem. Assume that X is a scheme with degree $\binom{n+d}{n}$ which is not a union of double points.

If $(n,d) \neq (4,5)$ and X has only 2 components with length less than or equal to n, we conclude by degenerating X to a union of double points.

If $(n,d) = (3,5)$ and X contains at least 7 double points, we specialize them on the trace and we conclude by the Castelnuovo sequence, since the residual contains 7 simple points. If X has less than 7 double points, then X has obviously at least 3 components with length less than or equal to 3. In this case we specialize a component with minimal length making it having residual 1, then we apply the construction of Lemma 5.2 on the remaining components and we conclude by the Castelnuovo sequence, since the residual contains at least a simple point.

If $(n,d) = (4,5)$ and X contains at least 14 double points, we specialize them on the trace and we conclude by the Castelnuovo sequence, since the residual contains 14 simple points. If X has less than 14 double points, then X has obviously at least 4 components with length less than or equal to 4. In this case we specialize a component with minimal length making it having residual 1, then we apply the construction of Lemma 5.2 on the remaining components and we conclude by the Castelnuovo sequence, since the residual contains at least a simple point.

If either $(n,d) = (3,6)$ or $(n,d) = (4,6)$ and X has at least 3 components with length less than or equal to 3, we conclude by Lemma 5.2 and by induction. □

We are now in position to give the general inductive argument which completes the proof of Theorem 1.1.

Given a scheme $X \subseteq \mathbf{P}^n$ of type $(m_1, \ldots, m_{n+1})$ and a fixed hyperplane $\mathbf{P}^{n-1} \subseteq \mathbf{P}^n$, we denote for any $1 \leq i \leq n+1$:

- by $m_i^{(1)}$ the number of component of length i completely contained in $\mathbf{P}^{n-1}$,
- by $m_i^{(2)}$ the number of component of length i supported on $\mathbf{P}^{n-1}$ and with residual 1 with respect to $\mathbf{P}^{n-1}$, and
- by $m_i^{(3)}$ the number of component of length i whose support does not lie in $\mathbf{P}^{n-1}$.

Obviously we have $m_i^{(1)} + m_i^{(2)} + m_i^{(3)} = m_i$, and $m_{n+1}^{(1)} = 0$, $m_1^{(2)} = 0$. We denote $t_i = m_i^{(1)} + m_{i+1}^{(2)}$, for $i = 1, \ldots, n+1$, $r_1 = m_1^{(3)} + \sum m_i^{(2)}$, and $r_i = m_i^{(3)}$ for $i = 2, \ldots, n+1$.

Theorem 5.6 *Fix the integers $n \geq 2$ and $d \geq 4$. A general zero-dimensional scheme $X \subset \mathbf{P}^n$ contained in a union of double points imposes independent conditions on $\mathcal{O}_{\mathbf{P}^n}(d)$ with the following exceptions*

- *$n = 2$, $d = 4$ and X is the union of 5 double points;*
- *$n = 3$ and either X is the union of 9 double points, or X is the union of 8 double points and a component of length 3;*
- *$n = 4$ and X is the union of 14 double points.*

Proof. We prove the statement by induction on n and d. In Lemma 5.1 we have proved the statement for $n = 2, d \geq 4$, in Lemma 5.3 and Lemma 5.4 for

$d = 4, 3 \leq n \leq 9$ and in Lemma 5.5 for $d = 5, n = 3, 4$ and $d = 6, n = 3, 4$. Then we need to prove the remaining cases. Assume $n \geq 3$ and in particular when $d = 4$ assume $n \geq 10$, and when $5 \leq d \leq 6$ assume $n \geq 5$.

The proof by induction is structured as follows:

- for $d = 4$ and $n \geq 10$, we assume that any scheme in $\mathbf{P}^n$ imposes independent conditions on $\mathcal{O}_{\mathbf{P}^{n-1}}(4)$. Recall that any scheme in $\mathbf{P}^n$ imposes independent conditions on $\mathcal{O}_{\mathbf{P}^n}(3)$ (by Theorem 4.1) and any scheme of degree greater than or equal to $(n+1)^2$ imposes independent conditions on $\mathcal{O}_{\mathbf{P}^n}(2)$ (by Theorem 3.2). Then we prove the statement for $d = 4, n \geq 10$;
- for $d \geq 5$ we assume that any scheme in $\mathbf{P}^a$ imposes independent conditions on $\mathcal{O}_{\mathbf{P}^a}(b)$ for $(a, b) \in \{(n-1, d), (n, d-1), (n, d-2)\}$ and we prove it for $(a, b) = (n, d)$.

It is enough to prove the statement for a scheme X with degree $\deg X = \binom{d+n}{n}$.

Let $X \subseteq \mathbf{P}^n$ be a scheme of type $(m_1, \ldots, m_{n+1})$ contained in a union of double points and suppose $\deg X = \sum i m_i = \binom{d+n}{n}$. Fix a hyperplane $\mathbf{P}^{n-1}$ in $\mathbf{P}^n$.

If there is a choice $m_i^{(1)}, m_i^{(2)}, m_i^{(3)}$ (and t_i, r_i as above) such that $\sum i t_i = \binom{d+n-1}{n-1}$, we can conclude by the Castelnuovo sequence and by induction

$$0 \to I_{X:\mathbf{P}^{n-1}}(d-1) \to I_X(d) \to I_{X \cap \mathbf{P}^{n-1}}(d).$$

Suppose now that such a decomposition is impossible, and fix a decomposition $m_i^{(1)}, m_i^{(2)}, m_i^{(3)}$ such that $\binom{d+n-1}{n-1} - \sum i t_i > 0$ is minimal and define

$$\varepsilon := \binom{d+n-1}{n-1} - \sum i t_i. \tag{8}$$

Obviously $0 < \varepsilon < n$ and $\varepsilon < \min\{i - 1 : m_i^{(3)} \neq 0\}$. By the minimality of our decomposition we have $m_1^{(3)} = m_2^{(3)} = 0$. Notice that by minimality of $\varepsilon > 0$ we have also $m_i^{(2)} = 0$ for all $i \neq n+1$.

Now let us define

$$\varepsilon_{n+1} = \min\{\varepsilon, m_{n+1}^{(3)}\}, \quad \varepsilon_n = \min\{\varepsilon - \varepsilon_{n+1}, m_n^{(3)}\}$$

and, for any $i = n-1, \ldots, 1$,

$$\varepsilon_i = \min\{\varepsilon - \sum_{k=i+1}^{n+1} \varepsilon_k, m_i^{(3)}\}.$$

Obviously we have $\varepsilon_1 = \varepsilon_2 = 0$ and $\sum_{i=3}^{n+1} \varepsilon_i = \varepsilon$.

Step 1: Let $\Gamma \subseteq \mathbf{P}^{n-1}$ be a general scheme of type $(\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{n+1}, 0)$ supported on a collection $\{\gamma_1, \ldots, \gamma_\varepsilon\} \subseteq \mathbf{P}^n$ of points and $\Sigma \subseteq \mathbf{P}^n$ a general scheme of type $(0, 0, m_3^{(3)} - \varepsilon_3, \ldots, m_{n+1}^{(3)} - \varepsilon_{n+1})$ supported at points which are not contained in $\mathbf{P}^{n-1}$.

By induction we know that

$$h_{\mathbf{P}^n}(\Gamma \cup \Sigma, d-1) = \min\left(\deg \Gamma \cup \Sigma, \binom{n+d-1}{n}\right)$$

where $\deg \Gamma \cup \Sigma = \sum (i-1)\varepsilon_i + \sum i(m_i^{(3)} - \varepsilon_i) = \sum i m_i^{(3)} - \varepsilon$.

From the definition of ε it follows that $\binom{n+d-1}{n} = m^{(2)}_{n+1} + \sum i m^{(3)}_i - \varepsilon$ and since of course $m^{(2)}_{n+1} \geq 0$, we obtain $h_{\mathbf{P}^n}(\Gamma \cup \Sigma, d-1) = \sum i m^{(3)}_i - \varepsilon$

Step 2: Now we want to add a collection Φ of $m^{(2)}_{n+1}$ simple points in $\mathbf{P}^{n-1}$ to the scheme $\Gamma \cup \Sigma$ and we want to obtain a $(d-1)$-independent scheme. From the previous step it is clear that $\dim I_{\Gamma\cup\Sigma}(d-1) = m^{(2)}_{n+1}$. Hence we have only to prove that there exist no hypersurfaces of degree $d-2$ through Σ. Notice that for $d \geq 5$ we have

$$\deg(\Sigma) = \sum i(m^{(3)}_i - \varepsilon_i) \geq \binom{n+d-2}{n} \tag{9}$$

and for $d = 4$ and $n \geq 10$ we have

$$\deg(\Sigma) = \sum i(m^{(3)}_i - \varepsilon_i) \geq (n+1)^2 \geq \binom{n+2}{n}. \tag{10}$$

Indeed by definition of ε, we have

$$\sum i(m^{(3)}_i - \varepsilon_i) = \binom{n+d-1}{n} + \varepsilon - \sum i\varepsilon_i - m^{(2)}_{n+1}$$

and since

$$\sum i\varepsilon_i - \varepsilon = \sum (i-1)\varepsilon_i \leq n\varepsilon \leq (n-1)n \quad \text{and} \quad m^{(2)}_{n+1} \leq \frac{1}{n}\binom{n+d-1}{n-1}$$

we obtain

$$\sum i(m^{(3)}_i - \varepsilon_i) \geq \binom{n+d-1}{n} - (n-1)n - \frac{1}{n}\binom{n+d-1}{n-1} =: S(n,d).$$

It is easy to check that for any $d \geq 5$ and $n \geq 3$ we have $S(n,d) > \binom{n+d-2}{n}$, which proves inequality (9). On the other hand one can also check that $S(n,4) > (n+1)^2$ for any $n \geq 10$, proving thus inequality (10).

Then by induction we know that Σ imposes independent conditions on $\mathcal{O}_{\mathbf{P}^n}(d-2)$, and so we get $\dim I_\Sigma(d-2) = 0$. Thus we obtain

$$h_{\mathbf{P}^n}(\Gamma \cup \Sigma \cup \Phi, d-1) = \sum i m^{(3)}_i - \varepsilon + m^{(2)}_{n+1} = \binom{n+d-1}{n}.$$

Step 3: Let us choose a family of general points $\{\delta^1_{t_1}, \ldots, \delta^\varepsilon_{t_\varepsilon}\} \subseteq \mathbf{P}^n$, with parameters $(t_1, \ldots, t_\varepsilon) \in K^\varepsilon$, such that for any $i = 1, \ldots, \varepsilon$ we have $\delta^i_0 = \gamma_i \in \mathbf{P}^{n-1}$ and $\delta^i_{t_i} \notin \mathbf{P}^{n-1}$ for any $t_i \neq 0$.

Now let us consider the following schemes:

- $\Delta_{(t_1,\ldots,t_\varepsilon)}$ a family of schemes of type $(\varepsilon_2, \ldots, \varepsilon_{n+1}, 0)$ supported at the points $\{\delta^1_{t_1}, \ldots, \delta^\varepsilon_{t_\varepsilon}\}$;
- Φ^2 of type $(0, \ldots, 0, m^{(2)}_{n+1})$, the union of double points supported on $\Phi \subset \mathbf{P}^{n-1}$;
- $\Psi \subseteq \mathbf{P}^{n-1}$ of type $(m^{(1)}_1, \ldots, m^{(1)}_n, 0)$ supported at general points of $\mathbf{P}^{n-1}$;
- $\Sigma \subseteq \mathbf{P}^n$, defined in Step 1, of type $(0, 0, m^{(3)}_3 - \varepsilon_3, \ldots, m^{(3)}_{n+1} - \varepsilon_{n+1})$.

By induction the scheme $(\Psi \cup \Phi^2|_{\mathbf{P}^{n-1}} \cup \Gamma) \subseteq \mathbf{P}^{n-1}$ has Hilbert function

$$h_{\mathbf{P}^{n-1}}(\Psi \cup \Phi^2|_{\mathbf{P}^{n-1}} \cup \Gamma, d) = \sum i m^{(1)}_i + n m^{(2)}_{n+1} + \varepsilon = \sum i t_i + \varepsilon = \binom{d+n-1}{n-1}$$

i.e. it is d-independent.

In order to prove that X imposes independent conditions on $\mathcal{O}_{\mathbf{P}^n}(d)$, it is enough to prove the following claim.

Claim: There exist $(t_1,\dots,t_\varepsilon)$ such that the scheme $\Delta_{(t_1,\dots,t_\varepsilon)}$ is independent with respect to the system $I_{\Psi\cup\Phi^2\cup\Sigma}(d)$.

Assume by contradiction that the claim is false. Then by Lemma 2.1 for any $(t_1,\dots,t_\varepsilon)$ there exist pairs $(\delta^i_{t_i},\eta^i_{t_i})$ for all $i=1,\dots,\varepsilon$, with $\eta^i_{t_i}$ a curvilinear scheme supported at $\delta^i_{t_i}$ and contained in $\Delta_{(t_1,\dots,t_\varepsilon)}$ such that

$$h_{\mathbf{P}^n}(\Psi\cup\Phi^2\cup\Sigma\cup\eta^1_{t_1}\cup\ldots,\eta^\varepsilon_{t_\varepsilon},d)<\binom{d+n}{n}-\sum(i-2)\varepsilon_i. \tag{11}$$

Let η^i_0 be the limit of $\eta^i_{t_i}$, for $i=1,\dots,\varepsilon$.

Suppose that $\eta^i_0\not\subset\mathbf{P}^{n-1}$ for $i\in F\subseteq\{1,\dots,\varepsilon\}$ and $\eta^i_0\subset\mathbf{P}^{n-1}$ for $i\in G=\{1,\dots,\varepsilon\}\setminus F$.

Given $t\in K$, let us denote $Z^F_t=\cup_{i\in F}(\eta^i_t)$ and $Z^G_t=\cup_{i\in G}(\eta^i_t)$. Denote by $\widetilde{\eta^i_0}$ for $i\in F$ the residual of η^i_0 with respect to $\mathbf{P}^{n-1}$ and by f and g the cardinalities respectively of F and G.

By the semicontinuity of the Hilbert function and by (11) we get

$$h_{\mathbf{P}^n}(\Psi\cup\Phi^2\cup\Sigma\cup Z^F_0\cup Z^G_t,d)\le h_{\mathbf{P}^n}(\Psi\cup\Phi^2\cup\Sigma\cup Z^F_t\cup Z^G_t,d)<\binom{d+n}{n}-\sum(i-2)\varepsilon_i.$$

On the other hand, by the semicontinuity of the Hilbert function there exists an open neighborhood O of 0 such that for any $t\in O$

$$h_{\mathbf{P}^n}(\Phi\cup\Sigma\cup(\cup_{i\in F}\widetilde{\eta^i_0})\cup Z^G_t,d-1)\ge h_{\mathbf{P}^n}(\Phi\cup\Sigma\cup(\cup_{i\in F}\widetilde{\eta^i_0})\cup Z^G_0,d-1)$$

Since the scheme $\Phi\cup\Sigma\cup(\cup_{i\in F}\widetilde{\eta^i_0})\cup Z^G_0$ is contained in $\Phi\cup\Sigma\cup\Gamma$, which is $(d-1)$-independent by Step 2, we have

$$h_{\mathbf{P}^n}(\Phi\cup\Sigma\cup(\cup_{i\in F}\widetilde{\eta^i_0})\cup Z^G_0,d-1)=m^{(2)}_{n+1}+\sum i(m^{(3)}_i-\varepsilon_i)+f+2g.$$

Since $\Psi\cup\Phi^2|_{\mathbf{P}^{n-1}}\cup(\cup_{i\in F}\gamma_i)$ is a subscheme of $\Psi\cup\Phi^2|_{\mathbf{P}^{n-1}}\cup\Gamma$, which is d-independent by Step 3, it follows that

$$h_{\mathbf{P}^{n-1}}(\Psi\cup\Phi^2|_{\mathbf{P}^{n-1}}\cup(\cup_{i\in F}\gamma_i),d)=\sum im^{(1)}_i+nm^{(2)}_{n+1}+f$$

Hence for any $t\in O$, by applying the Castelnuovo exact sequence to the scheme $\Psi\cup\Phi^2\cup\Sigma\cup Z^F_0\cup Z^G_t$, we get

$$h_{\mathbf{P}^n}(\Psi\cup\Phi^2\cup\Sigma\cup Z^F_0\cup Z^G_t,d)\ge$$

$$\ge h_{\mathbf{P}^n}(\Phi\cup\Sigma\cup(\cup_{i\in F}\widetilde{\eta^i_0})\cup Z^G_t,d-1)+h_{\mathbf{P}^{n-1}}(\Psi\cup\Phi^2|_{\mathbf{P}^{n-1}}\cup(\cup_{i\in F}\gamma_i),d)\ge$$

$$\ge(m^{(2)}_{n+1}+\sum i(m^{(3)}_i-\varepsilon_i)+f+2g)+(\sum im^{(1)}_i+nm^{(2)}_{n+1}+f)=$$

$$=\sum im_i-\sum i\varepsilon_i+2\varepsilon=\binom{d+n}{n}-\sum(i-2)\varepsilon_i,$$

contradicting (11). This completes the proof of the claim. □

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