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PREDICTABILITY, ENTROPY AND INFORMATION OF INFINITE TRANSFORMATIONS

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Abstract. We show that a certain type of conservative, ergodic,
measure preserving transformation always has a maximal zero entropy factor, generated by predictable sets. We also consider distribution asymptotics of information; e.g. for Boole's transformation, information is asymptotically mod-normal, a property shared by certain ergodic, probability preserving transformations with zero entropy.

§0 Introduction

Let $(X, \mathcal{B}, m, T)$ be a conservative, ergodic, measure preserving transformation and let $\mathcal{F} := \{F \in \mathcal{B} : \ m(F) < \infty\}$.

Call a set $A \in \mathcal{F}$ T-*predictable* if $A \in \sigma(\{T^{-n}A : \ n \geq 1\})$ and let $\mathcal{P} = \mathcal{P}_T := \{T\text{-predictable sets}\}$.

If $m(X) < \infty$, then Pinsker's theorem ([Pi]) says that

- $\mathcal{P}_T$ is the maximal, zero-entropy factor algebra

i.e. $\mathcal{P} \subset \mathcal{A}$ is a factor algebra (T-invariant, sub-σ-algebra), $h(T, \mathcal{P}) = 0$ and if $\mathcal{C} \subset \mathcal{A}$ is a factor algebra, with $h(S, \mathcal{C}) = 0$, then $\mathcal{C} \subseteq \mathcal{P}$. $\mathcal{P}$ is aka *Pinsker algebra* of $(X, \mathcal{B}, m, T)$.

When $(X, \mathcal{B}, m, T)$ is a conservative, ergodic,
measure preserving transformation with $m(X) = \infty$, the above statement fails and indeed $\sigma(\mathcal{P}) = \mathcal{B}$: Krengel has shown ([K2]) that:

- $\forall\ A \in \mathcal{F},\ \epsilon > 0,\ \exists\ B \in \mathcal{F},\ m(A \Delta B) < \epsilon$, a *strong generator* in the sense that $\sigma(\{T^{-n}B : \ n \geq 1\}) = \mathcal{B}$.

In particular, $\sigma(\mathcal{P}_T) = \mathcal{B}$ and it is not known if there is always a maximal, zero-entropy factor algebra (in case there is some zero-entropy factor algebra).

We define the class of *log lower bounded* conservative, ergodic,
measure preserving transformations (in §2).

These are quasi finite in the sense of [K1] (see §2, also for examples). A log lower bounded
conservative, ergodic, measure preserving transformation with some zero-entropy factor algebra has a maximal, zero-entropy factor algebra generated by a specified hereditary subring of predictable sets (see §5).

We obtain information convergence (in §4) for quasi finite transformations (extending [KS]).

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For quasi finite, pointwise dual ergodic transformations with regularly varying return sequences, we obtain (in §6) distributional convergence of information. Lastly, we construct probability preserving transformations with zero entropy with analogous distributional properties.

§1 Entropy

We recall the basic entropy theory of a probability preserving transformation $(\Omega, \mathcal{A}, P, S)$. Let $\alpha \subset \mathcal{A}$ be a countable partition.

- The *entropy* of α is $H(\alpha) := \sum_{a\in\alpha} P(a) \log \frac{1}{P(a)}$;
- the *S-join of α from k to ℓ* (for $k < \ell$) is

$$\alpha_k^\ell(S) := \{\bigcap_{j=k}^{\ell} S^{-j} a_j : \ a_k, a_{k+1}, \dots, a_\ell \in \alpha\}.$$

- By subadditivity, $\exists \ \lim_{n\to\infty} \frac{1}{n} H(\alpha_0^{n-1}(S)) =: h(S, \alpha)$ (the *entropy*[1] *of S with respect to α*).
- The *entropy of S* with respect to the *factor algebra* (S-invariant, σ-algebra), $\mathcal{C} \subset \mathcal{A}$ is $h(S, \mathcal{C}) := \sup_{\alpha\subset\mathcal{C}} h(S, \alpha)$.
- By the generator theorem, if α is a partition, then $h(S, \alpha) = h(S, \sigma(\{S^n\alpha : \ n \in \mathbb{Z}\}))$.
- **Convergence of information** is given by the celebrated Shannon- McMillan-Breiman theorem (see [S], [M], [B] respectively), the statement here being due to Chung [C]:

Let $(\Omega, \mathcal{A}, P, S)$ be an ergodic probability preserving transformation and let α be a partition with $H(\alpha) < \infty$, then

$$(\mathfrak{I}) \qquad P(\alpha_1^N(S)(x)) = e^{-nh(S,\alpha)(1+o(1))} \ \text{ a.s.}$$

where $x \in \alpha_1^N(S)(x) \in \alpha_1^N(s)$.

- We'll need **Abramov's formula** for the entropy of an induced transformation of an ergodic probability preserving transformation $(\Omega, \mathcal{A}, P, S)$:

$$h(S_A) = \tfrac{1}{P(A)} h(S) \ \forall \ A \in \mathcal{A}$$

where $S_A : A \to A$ is the *induced transformation* on A defined by

$$S_A x := S^{\varphi_A(x)} x, \ \varphi_A(x) := \min\{n \geq 1 : \ S^n x \in A\} \quad (x \in A).$$

- Abramov's formula can be proved using convergence of information (see [Ab] and §4 here).

[1] measure-theoretic

KRENGEL ENTROPY. Suppose that $(X, \mathcal{B}, m, T)$ is a conservative, ergodic, measure preserving transformation then using Abramov's formula (as shown in [K1])

$$m(A)h(T_A) = m(B)h(T_B) \quad \forall\ A, B \in \mathcal{F} := \{F \in \mathcal{B},\ 0 < m(F) < \infty\}.$$

Set $\underline{h}(T) := m(A)h(T_A), \quad (\text{any } A \in \mathcal{B},\ 0 < m(A) < \infty)$ – the *Krengel entropy of* T.

More generally, the *Krengel entropy of* T *with respect to the factor* (i.e. σ-finite, T-invariant sub-σalgebra) $\mathcal{C} \subset \mathcal{B}$ is

$$\underline{h}(T, \mathcal{C}) := m(A)h(T_A, \mathcal{C} \cap A) \quad (A \in \mathcal{C},\ 0 < m(A) < \infty).$$

- Another definition of entropy is given in [Pa]. It is shown in [Pa] that for *quasi finite* (see §2) conservative, ergodic, measure preserving transformations, the two entropies coincide.

§2 Quasifiniteness and Log lower boundedness

QUASIFINITENESS.

Let $(X, \mathcal{B}, m, T)$ be a conservative, ergodic, measure preserving transformation.

Recall from [K1] that a set $A \in \mathcal{F}$ is called *quasi finite* (qf) if $H_A(\rho_A) < \infty$ where $\rho_A := \{A \cap T^{-n}A \setminus \bigcup_{j=1}^{n-1} T^{-j}A :\ n \geq 1\}$ and that T is so called if $\exists$ such a set. As shown in proposition 7.1 in [K1],

- for $A \in \mathcal{F}$ quasi finite, $A \in \mathcal{P}_T \iff h(T_A, \rho_A) = 0$.

There are conservative, ergodic, measure preserving transformations which are not quasi finite. An unpublished example of such by Ornstein is mentioned in [K] and another (indirect) example is in [D-R].

Here we construct a conservative, ergodic, measure preserving transformation with no quasi finite extension. To do this we first establish a weak heredity property for the collection of quasi finite sets:

Proposition 2.0.
Suppose that $(X, \mathcal{B}, m, T)$ is a conservative, ergodic, quasi finite, measure preserving transformation, then $\forall\ F \in \mathcal{F},\ \exists\ A \in \mathcal{B} \cap F$ such that $m(A) > 0$ and such that each $B \in \mathcal{B} \cap A$ is quasi finite.

Proof We show first that

¶1 if $F \in \mathcal{F}$ is quasi finite, then $\forall\ \epsilon > 0,\ \exists\ A \in \mathcal{B} \cap F$ such that $m(F \setminus A) < \epsilon$ and such that each $B \in \mathcal{B} \cap A$ is quasi finite.

Proof By the $(\mathfrak{I})$, $\frac{1}{n} \log \frac{1}{m((\rho_F)_0^{n-1}(T_F))} \to nh(T_F, \rho_F)$ a.e. as $n \to \infty$. By Egorov's theorem, $\exists\ A \in \mathcal{B} \cap F$ such that $m(F \setminus A) < \epsilon$ and such that the convergence is uniform on A.

- For $B \in \mathcal{B} \cap A$, let $N_{n,B} := \#\{a \in (\rho_F)_0^{n-1}(T_F) :\ m(a \cap B) > 0\}$, then $N_{n,B} = e^{nh(T_F, \rho_F)(1+o(1))}$.
- Define $\psi : B \to \mathbb{N}$ by $\psi(x) := \min\{n \geq 1 :\ T_F^n x \in B\}$, then
$\int \psi dm = \sum_{n=1}^{\infty} m([\psi = n]) = m(F) < \infty$ (by Kac's formula);

- $\varphi_B(x) = \sum_{j=0}^{\psi(x)-1} \varphi_F(T_F^j x)$ whence

$$\rho_B \prec \gamma_B := \bigcup_{n=1}^{\infty} \{[\psi = n] \cap a : \ a \in (\rho_F)_0^{n-1}(T_F)\}.$$

Thus

$$\begin{aligned} H_{m_B}(\rho_B) &\le H_{m_B}(\gamma_B) \\ &= \sum_{n=1}^{\infty} m_B([\psi = n]) H_{m_{[\psi=n]}}((\rho_F)_0^{n-1}(T_F)) \\ &\le \sum_{n=1}^{\infty} m_B([\psi = n]) \log N_{n,B} < \infty \ \because \ \log N_{n,B} \sim nh(T_F, \rho_F). \quad \square \P 1 \end{aligned}$$

To complete the proof, let $F \in \mathcal{F}$. Suppose that $Q \in \mathcal{F}$ is quasi finite, then evidently so is $T^{-n}Q \ \forall \ n \ge 1$. By ergodicity, $\exists \ n \ge 1$ such that $m(F \cap T^{-n}Q) > 0$. By ¶1, $\exists \ G \in \mathcal{B} \cap T^{-n}Q$ such that $m(T^{-n}Q \setminus G) < \epsilon := \frac{m(F \cap T^{-n}Q)}{9}$ and such that each $B \in \mathcal{B} \cap A$ is quasi finite. The set $A = G \cap F$ is as required. $\square$

Example 2.1.

Let $(X, \mathcal{B}, m, T)$ be the conservative, ergodic, measure preserving transformation defined as in [F] by the cutting and stacking construction

$$B_0 = 1, \ B_n = \bigoplus_{k=1}^{N_n} B_{n-1} 0^{L_{n,k}}$$

where $N_n, \ L_{n,k} \ 1 \le k \le N_n$ satisfy

$$N_{n+1} \ge e^{nN_1 \dots N_n}, \quad L_{n,k+1} > \sum_{j=1}^{k} L_{n,k} + kh_{n-1},$$

where $h_n := |B_n|$.

Proposition 2.1.

No extension T of the conservative, ergodic, measure preserving transformation T_0 defined in example 2.1 is quasi finite.

Proof Suppose otherwise, that $(X, \mathcal{B}, m, T)$ is a (WLOG) conservative, ergodic extension of T_0 and that $F \in \mathcal{F}$ is quasi finite, then evidently so is $T^n F \ \forall \ n \ge 1$. By proposition 2.0 $\exists \ A \in \mathcal{B}, \ A \subset B_0$ quasi finite. We'll contradict this (and therefore the assumption that $\exists \ F \in \mathcal{F}$ quasi finite).

¶1 Write $B_n = \bigcup_{j=0}^{h_n-1} T^j b_n$ where $b_n \subset B_0, \ m(b_n) = \frac{1}{N_1 N_2 \dots N_n}$ and $B_n = \biguplus_{k=1}^{N_{n+1}} B_n^{(k)} = \biguplus_{k=1}^{N_{n+1}} T^{\kappa(n+1,k)} B_n^{(1)}$ where $\kappa(n+1,1) = 0$ and $\kappa(n+1,k) = (k-1)|B_n| + \sum_{j=1}^{k-1} L_{n+1,j}$ (i.e. the $B_n^{(k)} \ \ (1 \le k \le N_{n+1})$ are the subcolumns of B_n appearing in B_{n+1}).

¶2 For $n \ge 1$, let $\mathfrak{k}_n := \{0 \le j \le h_n - 1 : \ T^j b_n \subset B_0\}$, then

for $x \in b_n$, $\{T_{B_0}^k x\}_{k=0}^{N_1 N_2 \dots N_n - 1} = \{T^j x : \ j \in \mathfrak{k}_n\}$.

¶3 Fix $0 < \epsilon < \frac{1}{3}$ and let

$$b_{n,\epsilon} := \{x \in b_{n+1} : \ |\tfrac{1}{|\mathfrak{k}_{n+1}|} \sum_{k \in \mathfrak{k}_{n+1}} 1_A(T^k x) - m(A)| < \epsilon m(A)\}.$$

By ¶2 above, for $x \in b_{n+1}$,

$$\tfrac{1}{|\mathfrak{k}_{n+1}|} \sum_{k \in \mathfrak{k}_{n+1}} 1_A(T^k x) = \tfrac{1}{N_1 N_2 \dots N_{n+1}} \sum_{k=0}^{N_1 N_2 \dots N_n - 1} 1_A(T_{B_0}^k x)$$

and a standard argument using the ergodic theorem for T_{B_0} shows that $\exists\ M$ so that $m(b_{n,\epsilon}) > (1-\epsilon) m(b_{n+1}) \ \ \forall\ n \geq M$.

¶4 Fix $n \geq M$ and $x \in b_{n+1}$, let $\mathfrak{k}_{A,n,x} := \{k \in \mathfrak{k}_{n+1} : \ T^k x \in A\}$ and $A_{n,x} := \{T^j x\}_{j \in \mathfrak{k}_{A,n,x}}$, then for $x \in b_{n,\epsilon}$,

$$\#\{1 \leq k \leq N_{n+1} : \ A_{n,x} \cap B_n^{(k)} \neq \phi\} \geq (1-e) m(A) \tfrac{|\mathfrak{k}_{n+1}|}{h_n} = (1-\epsilon) m(A) N_{n+1}.$$

- For $n \geq M, \ x \in b_{n,\epsilon}$, write

$$\{1 \leq k \leq N_{n+1} : \ A_{n,x} \cap B_n^{(k)} \neq \phi\} =: \{\kappa_i(x) : \ 1 \leq i \leq \nu\}$$

where $\nu - 1 > (1-\epsilon) m(A) N_{n+1}$ and $\kappa_i(x) < \kappa_{i+1}(x) \ \forall\ i$.

- For $1 \leq i \leq \nu$, let $\mathfrak{k}_{A,n,x}^{(i)} := \{k \in \mathfrak{k}_{n+1} : \ T^k b_{n+1} \subset A_{n,x} \cap B_n^{(\kappa_i)}\}$ and let

$\underline{m}_i := \min \mathfrak{k}_{A,n,x}^{(i)}, \ \overline{m}_i := \max \mathfrak{k}_{A,n,x}^{(i)}$;

$y_i := \overline{m}_{i+1} - \underline{m}_i, \ \ (1 \leq i \leq \nu - 1)$. Note that

$y_i \leq \sum_{j=1}^{\kappa_i} L_{n+1,j} + \kappa_i h_n < L(n+1, \kappa_i + 1) \leq L(n+1, \kappa_{i+1}) \leq y_{i+1}$.

¶5 For $K \subset \mathfrak{k}_{n+1}$, let $a_K := \{x \in b_{n+1}, \ \mathfrak{k}_{A,n,x} = K\}$ and let

$$\beta_n := \{a_K : \ K \subset \mathfrak{k}_{n+1}\}, \ \ \alpha_n := \{\widehat{a} := \bigcup_{j \in \mathfrak{k}_n} T^j a : \ a \in \beta_n\}.$$

- For $a \in \beta_n, \ a \subset b_{n,\epsilon}, \ 1 \leq i \leq \nu - 1$,

$m(a \cap [\varphi_A = y_i(a)]) = \frac{m(a)}{N_1 \dots N_{n+1}}$.

- Thus

$$\begin{aligned} H(\rho_A) &\geq H(\rho_A \| \alpha_n) \\ &\geq \sum_{a \in \beta_n, \ a \subset b_{n,\epsilon}} m(a) \sum_{i=1}^{\nu-1} m([\varphi_A = y_i(a)]|a) \log \tfrac{1}{m([\varphi_A = y_i(a)]|a)} \\ &\geq m(\widehat{b}_{n,\epsilon}) \tfrac{(\nu-1)\log(N_{n+1})}{N_1 \dots N_{n+1}} \\ &\geq (1-\epsilon)^2 m(A) \tfrac{\log N_{n+1}}{N_1 N_2 \dots N_n} \\ &> (1-\epsilon)^2 m(A) n \uparrow \infty. \quad \square \end{aligned}$$

LOG LOWER BOUNDEDNESS.

For $(X, \mathcal{B}, m, T)$ a conservative, ergodic, measure preserving transformation; set

$$\mathcal{F}_{\log,T} := \{A \in \mathcal{B} : \ 0 < m(A) < \infty, \ \int_A \log \varphi_A dm < \infty\}.$$

- Note that

$\mathcal{F}_{\log,T} \subset \{\texttt{quasi finite sets}\}$;
because $p_n \geq 0, \ \sum_{n=1} \log np_n < \infty \implies \sum_{n=1} p_n \log \frac{1}{p_n} < \infty$.

- Call T log-*lower bounded* (LLB) if $\mathcal{F}_{\log,T} \neq \emptyset$.

Proposition 2.2.

(i) *T is LLB iff $\frac{1}{\log n} \sum_{k=0}^{n-1} f \circ T^n \to \infty$ a.e. as $n \to \infty$ for some and hence all $f \in L^1_+$;*

(ii) T is not LLB iff $\liminf_{n\to\infty} \frac{1}{\log n} \sum_{k=0}^{n-1} f \circ T^n = 0$ a.e. for some and hence all $f \in L^1_+$;

(iii) *If $(X, \mathcal{B}, m, T)$ is LLB and $\mathcal{C} \subset \mathcal{B}$ is a factor, then $\mathcal{C} \cap \mathcal{F}_{\log,T} \neq \emptyset$.*

(iv) *$\mathcal{F}_{\log,T}$ is a hereditary ring.*

Proof Statements (i) and (ii) follow from theorem 2.4.1 in [A] and (iii) follows from these. We prove (iv).

Suppose that $A \in \mathcal{F}_{\log,T}, \ B \in \mathcal{B}, \ B \subset A$, then $\varphi_B(x) = \sum_{k=0}^{\psi(x)-1} \varphi_A(T_A^k x) \quad (x \in B)$ where $\psi : B \to \mathbb{N}, \ \psi(x) := \min\{n \geq 1 : \ T_A^n x \in B\}$.

By Kac formula,

$$\int_B \sum_{k=0}^{\psi-1} f \circ T_A^k dm = \int_A f dm \ \forall \ f \in L^1(m).$$

To see that $B \in \mathcal{F}_{\log,T}$, we use this and $\log(k+\ell) \leq \log(k) + \log(\ell)$:

$$\begin{aligned} \int_B \log \varphi_B dm &= \int_B \log(\sum_{k=0}^{\psi-1} \varphi_A \circ T_A^k) dm \\ &\leq \int_B \sum_{k=0}^{\psi-1} \log(\varphi_A \circ T_A^k) dm \\ &= \int_A \log \varphi_A dm < \infty. \end{aligned}$$

Suppose that $A, B \in \mathcal{F}_{a,T}$, then $\varphi_{A\cup B} \leq 1_A \varphi_A + 1_B \varphi_B$ whence

$$\begin{aligned} \int_{A\cup B} \log(\varphi_{A\cup B}) dm &= \int_A \log(\varphi_{A\cup B}) dm + \int_B \log(\varphi_{A\cup B}) dm \\ &\leq \int_A \log(\varphi_A) dm + \int_B \log(\varphi_B) dm \\ &< \infty. \quad \square \end{aligned}$$

§3 Examples of LLB transformations

POINTWISE DUAL ERGODIC, TRANSFORMATIONS.

Proposition 3.1.
Let $(X, \mathcal{B}, m, T)$ be a pointwise dual ergodic, conservative, ergodic, measure preserving transformation, then T is LLB $\iff \sum_{n=1}^{\infty} \frac{1}{na_n(T)} < \infty$.

Proof
$\Rightarrow$) By theorem 3.8.1 in [A], $m(\bigcup_{k=0}^{n} T^{-k}A) \gg \frac{n}{a_n(T)}\ \forall\ A \in \mathcal{F},\ \ m(A) > 0$. Thus, if T is LLB then $\sum_{n=1}^{\infty} \frac{1}{na_n(T)} < \infty$.
$\Leftarrow$) Let $A \in \mathcal{F}$ be a *uniform set* in the sense that for some $f \in L^1(m)_+$

$$\tfrac{1}{a_n(T)} \sum_{k=0}^{n-1} \widehat{T}^k f \to \int_X f dm \ \text{ uniformly on } \ A.$$

By lemma 3.8.5 in [A],

$$\int_A (\varphi_A \wedge n) dm = m(\bigcup_{k=0}^{n} T^{-k}A) \asymp \tfrac{n}{a_n(T)}$$

whence

$$A \in \mathcal{F}_{\log} \iff \sum_{n=1}^{\infty} \tfrac{m(\bigcup_{k=0}^{n} T^{-k}A)}{n} < \infty \iff \sum_{n=1}^{\infty} \tfrac{1}{na_n(T)} < \infty. \quad \square$$

Remarks.
1) For example, the simple random walk on $\mathbb{Z}$ is LLB ($\because\ a_n(T) \propto \sqrt{n}$); whereas the simple random walk on $\mathbb{Z}^2$ is not LLB ($\because\ a_n(T) \propto \log n$).
2) It is not known whether the simple random walk on $\mathbb{Z}^2$ is quasi finite, or even has a factor with finite entropy.

Example 3.2.
There is a quasi finite, conservative, ergodic, Markov shift $(X, \mathcal{B}, m, T)$ with $a_n(T) \asymp \sqrt{\log n}$.

- By proposition 3.1, T is not LLB.

Proof Let $f_{4^{4^n}} := \frac{1}{2^n}\ \ n \geq 1$ and $f_k := 0\ \forall\ k \in \mathbb{N} \setminus 4^{4^{\mathbb{N}}}$, then $f \in \mathcal{P}(\mathbb{N})$.

- Let $\Omega := \mathbb{N}^{\mathbb{Z}}$ and let $P = f^{\mathbb{Z}} \in \mathcal{P}(\Omega, \mathcal{B}(\Omega))$ be product measure, then $(\Omega, \mathcal{B}(\Omega), P, S)$ in an ergodic, probability preserving transformation where $S : \Omega \to \Omega$ is the shift.
- Define $\varphi : \Omega \to \mathbb{N}$ by $\varphi(\omega) := \omega_0$ and let $(X, \mathcal{B}, m, T)$ be the tower over $(\Omega, \mathcal{B}(\Omega), P, S)$ with height function φ.
- It follows that $(X, \mathcal{B}, m, T)$ is a conservative, ergodic, Markov shift with $a_n(T) \asymp \sum_{k=0}^{n} u_k$ where u is defined by the renewal equation: $u_0 = 1,\ u_n = \sum_{k=1}^{n} f_k u_{n-k}$.
- To see that $(X, \mathcal{B}, m, T)$ is quasi finite, we check that Ω is quasi finite. Indeed

$$H_\Omega(\rho_\Omega) = \sum f_k \log \tfrac{1}{f_k} = \sum^{\infty} \tfrac{n \log 2}{2^n} < \infty.$$

- To estimate $a_n(T)$, recall that by (e.g.) lemma 3.8.5 in [A], $a_n(T) \asymp \frac{n}{L(n)}$ where

$$L(n) := m(\bigcup_{k=0}^{n} T^{-k}\Omega) = \sum_{k=0}^{n}\sum_{\ell=k+1}^{\infty} f_\ell.$$

Now,

$$\sum_{\ell=k+1}^{\infty} f_\ell = \sum_{n>\log_4\log_4 k} \tfrac{1}{2^n} \asymp \tfrac{1}{2^{\log_4\log_4 k}} = \tfrac{1}{\sqrt{\log_4 k}}.$$

Thus $L(n) \asymp \frac{n}{\sqrt{\log n}}$ and $a_n(T) \asymp \sqrt{\log n}$. $\square$

THE HAJIAN-ITO-KAKUTANI TRANSFORMATIONS.

- Let $\Omega = \{0,1\}^{\mathbb{N}}$, $\ell(\omega) := \min\{n \geq 1 :\ \omega_n = 0\}$ and let $\tau : \Omega \to \Omega$ be the *adding machine* defined by

$$\tau(1,\dots,1,0,\omega_{\ell(\omega)+1},\dots) := (0,\dots,0,1,\omega_{\ell(\omega)+1},\dots).$$

For $p \in (0,1)$, define $\mu_p \in \mathcal{P}(\Omega)$ by $\mu_p([a_1,\dots,a_n]) := p_{a_1}\dots p_{a_n}$ where $p_0 := 1-p$, $p_1 := p$. It follows that $(\Omega, \mathcal{A}, \mu_p, \tau)$ is an ergodic, nonsingular transformation with $\frac{d\mu_p\circ\tau}{d\,\mu_p} = (\frac{1-p}{p})^\phi$ where $\phi := \ell - 2$.

Now let $X := \Omega \times \mathbb{Z}$, $T := \tau_\phi$. For $p \in (0,1)$, define $m_p \in \mathfrak{M}(X)$ by $m_p(A \times \{n\}) := \mu_p(A)(\frac{1-p}{p})^{-n}$.

As shown in [HIK] (see also [A]) $T_p = (X, \mathcal{B}, m_p, T)$ is a conservative, ergodic, measure preserving transformation (aka the *Hajian-Ito-Kakutani* transformation). The entropy is given by $\underline{h}(T_p) = h((T_p)_{\Omega\times\{0\}}) = 0$ by [MP] since $(T_p)_{\Omega\times\{0\}}$ is the Pascal adic transformation.

Proposition 3.3.
$(X, \mathcal{B}, m_p, T)$ is LLB $\forall\ 0 < p < 1$.

Proof

As in the proof of proposition 5.1 in [A1],

$$\sum_{k=0}^{2^n-1} 1_{\Omega\times\{0\}}\circ T^k(x,0) = \#\{0 \leq k \leq 2^n-1 :\ \sum_{j=0}^{k-1}\phi(\tau^j x) = 0\}$$
$$\geq \#\{0 \leq K \leq n-1 :\ \sum_{j=0}^{2^K-1}\phi(\tau^j x) = 0\}$$

Now $\sum_{j=0}^{2^K-1}\phi(\tau^j x) = \phi(S^K x)$ where $S : \Omega \to \Omega$ is the shift, and so

$$\sum_{k=0}^{2^n-1} 1_{\Omega\times\{0\}}\circ T^k(x,0) \geq \#\{0 \leq K \leq n-1 :\ \phi(S^K x) = 0\} \sim (1-p)n$$

for μ_p-a.e. $x \in \Omega$ by Birkhoff's theorem for the
ergodic, probability preserving transformation $(\Omega, \mathcal{B}(\Omega), \mu_p, S)$. The LLB property now follows from proposition 2.1. $\square$

- Let $\mathfrak{G}$ be the Polish group of measure preserving transformations of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m_{\dots})$ equipped with the weak topology

Proposition 3.4.

The collection of LLB measure preserving transformation's is meagre in $\mathfrak{G}$.

Proof Let

$$\mathcal{L} := \{T \in \mathfrak{G} : \ \exists\ n_k \to \infty,\ \tfrac{S_{n_k}(f)}{\log n_k} \to 0 \text{ a.e. } \ \forall\ f \in L^1\}$$

where $S_n(f) = S_n^T(f) := \sum_{j=0}^{n-1} f \circ T^j$.

By proposition 2.1, it suffices to show that $\mathcal{L}$ is a dense G_δ set in $\mathfrak{G}$.

By example 3.2, $\exists$ a conservative, ergodic, measure preserving transformation $T \in \mathcal{L}$. $\mathcal{L}$ is conjugacy invariant, and so dense in $\mathfrak{G}$ by the conjugacy lemma (e.g. 3.5.2 in [A]).

To see that $\mathcal{L}$ is a G_δ set, let

- $P \sim m$ be a probability;
- fix $\{A_n : \ n \in \mathbb{N}\} \subset \mathcal{F} := \{A \in \mathcal{B} : m(A) < \infty\}$ so that $\sigma(\{A_n : \ n \in \mathbb{N}\}) = \mathcal{B}$ and let

$$\mathcal{L}' := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \bigcap_{\nu=1}^{k} \{T \in \mathfrak{G} : \ P([S_n(1_{A_\nu}) > \tfrac{1}{k} \log n]) < \frac{1}{2^k}\},$$

then $\mathcal{L}'$ is a G_δ. We claim $\mathcal{L}' = \mathcal{L}$.

Evidently,

$$\mathcal{L}' = \{T \in \mathfrak{G} : \ \exists\ n_k \to \infty \text{ such that } \tfrac{S_{n_k}(1_{A_\nu})}{\log n_k} \to 0 \text{ a.e. } \forall\ \nu \geq 1\}$$

whence $\mathcal{L}' \supset \mathcal{L}$.

Now suppose that $T \in \mathcal{L}'$, that $\frac{S_{n_k}(1_{A_\nu})}{\log n_k} \to 0$ a.e. $\forall\ \nu \geq 1\}$ and let $f \in L^1$. Evidently $\frac{S_n(f)}{\log n} \to 0$ a.e. on $\mathfrak{D}$, the dissipative part of T. The conservative part of T:

$$\mathfrak{C} = \bigcup_{\nu=1}^{\infty} \hat{A}_\nu \ \texttt{where}\ \hat{A}_\nu := [\sum_{n=1}^{\infty} 1_{A_\nu} \circ T^n = \infty].$$

By Hopf's theorem, $\frac{S_n(f)(x)}{S_n(1_{A_\nu})(x)} \to h_\nu(f)$ a.e. on $A_\nu\ \forall\ \nu \geq 1$ where $h_\nu(f) \circ T = h_\nu(f)$ and $\int_{A_\nu} h_\nu(f) dm = \int_X f dm$, whence, a.e. on $\hat{A}_\nu$,

$$\frac{S_{n_k}(f)}{\log n_k} = \frac{S_{n_k}(f)}{S_{n_k}(1_{A_\nu})} \cdot \frac{S_{n_k}^T(1_{A_\nu})}{\log n_k} \to 0. \quad \square$$

§4 `Information convergence`

Let $(X, \mathcal{B}, m, T)$ be a conservative, ergodic, measure preserving transformation.

- A countable partition $\xi \subset \mathcal{B}$ is called *cofinite* if $\exists\ A = A_\xi \in \mathcal{F}$ with $A^c \in \xi$. We call A^c the *cofinite atom* of ξ and A the *(finite) core* of ξ..
- If $\xi \subset \mathcal{B}$ is cofinite, then $\xi_k^\ell(T)$ is also cofinite, with core $A_{\xi_k^\ell(T)} = \bigcup_{j=k}^{\ell} T^{-j} A$.

The T-process generated by a cofinite partition ξ restricted to its core A is given

Krengel's formula [K1]:

$$(\mathfrak{K}) \qquad \xi_1^{\varphi_n(x)}(T)(x) \;=\; (\rho_A \vee ((\xi \cap A) \vee \rho_A)_1^n(T_A))(x) \text{ for a.e. } x \in A$$

where for $x \in X$, α a partition of X, $\alpha(x)$ is defined by $x \in \alpha(x) \in \alpha$;
$\varphi_n(x) := \sum_{k=0}^{n-1} \varphi_A(T_A^k x)$; and
$\rho_A := \{A \cap T^{-n}A \setminus \bigcup_{k=1}^{n-1} A : \; n \in \mathbb{N}\}$.

- A cofinite partition $\xi \subset \mathcal{B}$ is called *quasi-finite* (qf) if $A = A_\xi$ is quasi finite and $H_A(\xi) < \infty$.
- Note that ξ quasi finite $\Rightarrow H_{A_\xi}(\xi \vee \rho_{A_\xi} \vee T_{A_\xi}\rho_{A_\xi}) < \infty$.

Convergence of information for quasi finite partitions.

Proposition 4.1 (c.f. [KS]).
Let $(X, \mathcal{B}, m, T)$ be a conservative, ergodic, measure preserving transformation, let $\xi \subset \mathcal{B}$ be a quasi finite partition and let $p \in L^1(m)$, $p > 0$, $\int_X p dm = 1$, then for a.e. $x \in X$,

$$\tfrac{1}{s_n p(x)} \log \tfrac{1}{m(\xi_1^n(T)(x))} \to \underline{h}(T, \xi)$$

where $s_n p(x) := \sum_{k=0}^{n-1} p(T^k x)$.

Proof Let A be the core of ξ and set $\varsigma := (\xi \cap A) \vee \rho_A$, then by $(\mathfrak{K})$

$$\varsigma_0^{s_n(x)}(T_A)(x) \subseteq \xi_1^n(T)(x) \subseteq \varsigma_1^{s_n(x)-1}(T_A)(x) \;\text{ a.e. } x \in A$$

where $x \in \xi(x) \in \xi$, $s_n := s_n 1_A$.

- By $(\mathfrak{I})$, for T_A, for a.e. $x \in A$, $\log \frac{1}{m(\varsigma_1^N(T_A)(x))} \sim N h(T_A, \varsigma)$, whence

$$\begin{aligned} \log \tfrac{1}{m(\xi_1^n(T)(x))} &\sim \log \tfrac{1}{m(\varsigma_1^{s_n(x)}(T_A)(x))} \\ &\sim s_n(x) h(T_A, \varsigma) \\ &\sim s_n p(x) m(A) h(T_A, \varsigma) \\ &= s_n p(x) \underline{h}(T, \xi). \end{aligned}$$

We obtain convergence a.e. on $\bigcup_{k=0}^N T^{-k}A$ by substituting $\xi_1^N(T)$ for ξ; whence convergence a.e. on X as $\bigcup_{k=0}^N T^{-k}A \uparrow X$. $\square$

- Abramov's formula is proved analogously in case $(X, \mathcal{B}, m, T)$ is an ergodic, probability preserving transformation. As in [Ab]:

$$h(T, \xi) \overset{(\mathfrak{I})}{\leftarrow} \tfrac{1}{n} \log \tfrac{1}{m(\xi_1^n(T)(x))} \approx \tfrac{1}{n} s_n(x) h(T_A, \varsigma) \overset{\text{Birkhoff's PET}}{\to} m(A) h(T_A, \varsigma).$$

§5 `Pinsker algebra`

Let $(X, \mathcal{B}, m, T)$ be a LLB, conservative, ergodic,
measure preserving transformation.

Define

$$\mathcal{F}_\Pi := \{A \in \mathcal{F}_{\log,T} : \; A \in \sigma(\{T^{-k}A : \; k \geq 1\})\} = \mathcal{P} \cap \mathcal{F}_{\log,T}.$$

In this section, we show that (in case $\mathcal{F}_\Pi \neq \emptyset$) $\mathcal{B}_\Pi := \sigma(\mathcal{F}_\Pi)$ is the maximal zero entropy factor of T.

To do this, we'll need

Krengel's predictability lemma. *[K1]:*
Let $(X, \mathcal{B}, m, T)$ be a quasi finite, conservative, ergodic, measure preserving transformation, let $\xi \subset \mathcal{B}$ be a quasi finite partition with core A, and let $\zeta = \xi \cap A$, then

$$\xi \subset \xi_1^\infty(T) \mod m \iff h(T_A, \zeta \vee \rho_A) = 0.$$

In particular

$$A \in \sigma(\{T^{-n}A : \ n \geq 1\}) \iff h(T_A, \rho_A) = 0.$$

- For $F \in \mathcal{F}$, set

$$\mathcal{P}_F = \mathcal{P}_{T_F} := \{A \in \mathcal{B} \cap F : \ A \in \sigma(\{T_F^{-k}A : \ k \geq 1\})\}.$$

By Pinsker's theorem ([Pi]),

- $\mathcal{P}_F$ is a T_F-factor algebra of subsets of F, $h(T_F, \mathcal{P}_F) = 0$ and
- if $\mathcal{A} \subset \mathcal{B} \cap F$ is another T_F-factor algebra of subsets of F with $h(T_F, \mathcal{A}) = 0$, then $\mathcal{A} \subset \mathcal{P}_F$.

Theorem 5.1.

(i) *$\mathcal{F}_\Pi$ is a ring and $\mathcal{F}_\Pi \cap F = \mathcal{P}_F \ \forall \ F \in \mathcal{F}_\Pi$.*

(ii) *If $\mathcal{F}_\Pi \neq \emptyset$, then $\sigma(\mathcal{F}_\Pi)$ is the maximal factor of zero entropy.*

`Proof.`

¶1 Let $A \in \mathcal{F}_{\log}$. By Krengel's predictability lemma, $F \in \mathcal{F}_\Pi$ iff $h(T_F, \rho_F) = 0$.
Thus, $F \in \mathcal{F}_\Pi$ iff $\exists$ a factor $\mathcal{B}_0$ with $F \in \mathcal{B}_0$ and $\underline{h}(T, \mathcal{B}_0) = 0$.

¶2 Next, fix $F \in \mathcal{F}_\Pi$. We claim that $\rho_F \subseteq \mathcal{P}_F$. This is because $F \in \mathcal{F}_\Pi \ \Rightarrow \ h(T_F, \rho_F) = 0$.

¶3 We now show that $\mathcal{P}_F \subseteq \mathcal{F}_\Pi \cap F \ \forall \ F \in \mathcal{F}_\Pi$.

Proof: Fix $F \in \mathcal{F}_\Pi$ and let $\mathcal{B}_0 := \sigma\{T^n A : \ n \in \mathbb{Z}, \ A \in \mathcal{P}_F\}$, then $\mathcal{B}_0$ is a factor, $F \in \mathcal{B}_0$ and $\mathcal{B}_0 \cap F = \mathcal{P}_F$. Thus $\underline{h}(T, \mathcal{B}_0) = h(T_F, \mathcal{P}_F) = 0$ and by ¶1 $\mathcal{P}_F \subseteq \mathcal{F}_\Pi \cap F$.☺

¶4 Now we claim that $A, B \in \mathcal{F}_\Pi \ \Rightarrow \ A \cup B \in \mathcal{P}_F$.

Proof: Set $C := A \cup B$, then $C \in \mathcal{F}_{\log,T}$. Set $\zeta := \{A \cap B, A \setminus B, B \setminus A\}$ and $\xi := \zeta \cup \{C^c\}$. By $(\mathfrak{K})$,

$$\xi_1^\infty(T) \cap C = \rho_C \vee (\zeta \vee \rho_C)_1^\infty(T_C).$$

By assumption, $\zeta \subset \xi_1^\infty(T) \cap C$, whence also $\rho_C \subset \xi_1^\infty(T) \cap C$. Thus

$$\zeta \vee \rho_C \subset \rho_C \vee (\zeta \vee \rho_C)_1^\infty(T_C); \quad \therefore \ \zeta \vee \rho_C \vee T_C\rho_C \subset (\zeta \vee \rho_C \vee T\rho_C)_1^\infty(T_C),$$

and (using $H_C(\zeta \vee \rho_C \vee T_C\rho_C) < \infty$) we have

$$h(T_C, \rho_C) \leq h(T_C, \zeta \vee \rho_C \vee T_C\rho_C) = 0$$

whence $C \subset \sigma(\{T^{-k}C : \ k \geq 1\})$ and $C \in \mathcal{F}$. ☺

¶5 Now we show that $\mathcal{F}_\Pi$ is a ring by proving that $A, B \in \mathcal{F}_\Pi \ \Rightarrow \ \zeta := \{A \cap B, A \setminus B, B \setminus A\} \subset \mathcal{F}_\Pi$.

Proof. By ¶3, it suffices to show that $\zeta \subset \mathcal{P}_C$ where $C := A \cup B$. To see this, fix $a \in \zeta$, then

$$h(T_C, \{a, C \setminus a\}) \le h(T_C, \zeta) \le h(T_C, \zeta \vee \rho_C \vee T_C \rho_C) = 0$$

(as above) and $a \in \mathcal{P}_C$. ☺

¶6 To complete the proof of (i), we show that $\mathcal{F}_\Pi \cap F \subseteq \mathcal{P}_F \ \forall \ F \in \mathcal{F}_\Pi$.

Proof: Fix $F \in \mathcal{F}_\Pi, \ A \in \mathcal{F}_\Pi \cap F$. Let $\zeta := \{A, F \setminus A\}, \ \xi := \zeta \cup \{F^c\}$.

By the ring property, $A \in \mathcal{F}_\Pi$, whence $\xi \subset \xi_1^\infty(T) \mod m$. By proposition 4, $h(T_F, \zeta \vee \rho_F) = 0$, whence

$$h(T_F, \zeta) \le h(T_F, \zeta \vee \rho_F) = 0$$

and $A \in \mathcal{P}_F$.☺

¶7 To see (ii), fix $F \in \mathcal{F}_\Pi$, then by (i), $\mathcal{F}_\Pi \cap F = \mathcal{P}_F = \mathcal{F}_\Pi \cap F \cap F$ whence $\underline{h}(T, \sigma(\mathcal{F}_\Pi)) = m(F) h(T_F, \mathcal{P}_F) = 0$ and if $\mathcal{C} \subset \mathcal{B}$ is a factor with $\underline{h}(T, \mathcal{C}) = 0$, then by ¶1, $\mathcal{C} \cap \mathcal{F}_{\log} \subset \mathcal{F}_\Pi$, whence $\mathcal{C} \subset \sigma(\mathcal{F}_\Pi)$. □

§6 Asymptotic distribution of information

Let $(X, \mathcal{B}, m, T)$ be a pointwise dual ergodic measure preserving transformation and assume that the return sequence $a_n = a_n(T)$ is regularly varying with index $\alpha \ \ (\alpha \in [0, 1])$, then by the Darling-Kac theorem,

$$(\text{☿}) \qquad \tfrac{1}{a_n} S_n^T(f) \xrightarrow{\mathfrak{d}} \int_X f dm \cdot X_\alpha \ \ \texttt{as} \ n \to \infty \ \forall \ f \in L^1(m)_+$$

where

- X_α is a Mittag-Leffler random variable of order α normalised so that $E(X_\alpha) = 1$; and
- $F_n \xrightarrow{\mathfrak{d}} Y$ means

$$\int_X G(F_n) dP \to E(G(Y)) \ \forall \ P \in \mathcal{P}(X, \mathcal{B}), \ P \ll m, \ \ G \in C([0, \infty]).$$

Note that $X_1 \equiv 1$, the density of X_0 is given by $f_{X_0}(t) = e^{-t} 1_{\mathbb{R}_+}(t)$ and for $\alpha \in (0, 1)$, $X_\alpha = \frac{1}{Y_\alpha^\alpha}$ where $E(e^{-tY_\alpha}) = e^{-ct^\alpha}$ (some $c = c_\alpha > 0$). In particular $X_{\frac{1}{2}} = \mathcal{N}$ where $\mathcal{N}$ is a centered Gaussian random variable on $\mathbb{R}$ with $E(|\mathcal{N}|) = 1$.

Proposition 6.1. *Suppose that $(X, \mathcal{B}, m, T)$ is a quasi finite, pointwise dual ergodic measure preserving transformation and assume that the return sequence $a_n = a_n(T)$ is regularly varying with index $\alpha \ \ (\alpha \in [0, 1])$.*

If $\xi \subset \mathcal{B}$ is quasi finite, then

$$\tfrac{1}{a_n(T)} \log \tfrac{1}{m(\xi_1^n(T)(x))} \xrightarrow{\mathfrak{d}} \underline{h}(T, \xi) X_\alpha$$

as $n \to \infty$.

Proof. This follows from proposition 4.1 and (☿). □

Example 6.2: `Boole's transformation.`
Let $(X, \mathcal{B}, m, T)$ be given by $X = \mathbb{R},\ m =$ Lebesgue measure and $Tx = x - \frac{1}{x}$, then T (see [A]) is a pointwise dual ergodic, measure preserving transformation with $a_n(T) \sim \frac{\sqrt{2n}}{\pi}$, so $\mathcal{F}_\Pi \neq \emptyset$ and T is LLB, whence quasi finite.

- By Proposition 6.1, if $\xi \subset \mathcal{B}$ is quasi finite, then

$$\tfrac{1}{a_n(T)} \log \tfrac{1}{m(\xi_1^n(T)(x))} \xrightarrow{\mathfrak{d}} \underline{h}(T, \xi)|\mathcal{N}|$$

as $n \to \infty$.

Example 6.3: `A skew product probability preserving transformation.`
Let R be an irrational rotation of the circle $\mathbb{T}$.
Let $f \in L^2(\mathbb{T})$ so that $\frac{1}{\sqrt{n}} \max_{1 \le k \le n} |f_k| \xrightarrow{\mathfrak{d}} |\mathcal{N}|$ (where $f_k := \sum_{j=0}^{k-1} f \circ R^j$). Existence of such $f \in L^2(\mathbb{T})$ is shown in [V]
Let $(Y, \mathcal{C}, \mu, S)$ be the 2-shift with symmetric product measure, $h : Y \to \mathbb{R}$ be Hölder continuous so that the *flow under* h (Y^h, q, S^h) is mixing; where

$$Y^h := \{(y, s) :\ y \in Y,\ s \in [0, h(y))\},\ q := \mu \times \lambda,\ S_t^h(y, s) := (S^n y, s + t - h_n(y))$$

where $0 \le s + t - h_n(y) < h(S^n y),\ h_n := \sum_{j=0}^{n-1} h \circ S^j$.

Define the probability preserving transformation $(X, \mathcal{B}, m, T)$ by

$$X := \mathbb{T} \times Y^h,\ m = m_{\mathbb{T}} \times q,\ T(x, (y, s)) := (R(x), S_{f(x)}^h(y, s)).$$

Proposition 6.4. *$(X, \mathcal{B}, m, T)$ is ergodic, $h(T) = 0$ and $\exists$ finite T-generators $\xi \subset \mathcal{B}$ so that*

$$(\maltese) \qquad \tfrac{1}{\sqrt{n}} \log \tfrac{1}{m(\xi_0^{n-1}(T)(x,y,s))} \xrightarrow[n \to \infty]{\mathfrak{d}} 2|\mathcal{N}|h(S_1^h).$$

Proof

`Ergodicity of` T: The R-cocycle generated by $f : \mathbb{T} \to \mathbb{R}$ is not a coboundary and the flow S^h is mixing whence by [LL] (see also corollary 3.2 of [AL]), T is ergodic.
$h(T) = 0$ follows from the Abramov-Rokhlin formula (in [AR]).
To see $(\maltese)$ let ξ be a finite T-generator of form $\xi = \alpha \times \beta$ where each $a \in \alpha$ is an interval and each $b \in \beta$ is of form $b = \{(y, s) \in Y^h :\ y \in c\},\ c \in \gamma,\ \gamma \subset \mathcal{C}$ a partition of Y. For such ξ and any $\epsilon > 0$, for sufficiently large $n = n(x, y, s)$:

$$\begin{aligned} \alpha_0^{n-1}(R)(x) \times \{(z, u) :\ z \in \gamma_{-m_n(x)(1+\epsilon)}^{m_n(x)(1+\epsilon)}(S)(y),\ 0 \le z \le h(z)\} &\subseteq \xi_0^{n-1}(T)(x, y, s) \\ &\subseteq \mathbb{T} \times \{(z, u) :\ z \in \gamma_{-m_n(x)(1-\epsilon)}^{m_n(x)(1-\epsilon)}(S)(y),\ 0 \le z \le h(z)\} \end{aligned}$$

where $m_n(x) := \frac{\max_{1 \le k \le n} |f_k(x)|}{\int_Y h d\mu}$.

Using $(\beth)$ for S and the fact that $\log \# \alpha_0^{n-1}(R) = O(\log n)$ as $n \to \infty$, we obtain

$$\begin{aligned} \log \tfrac{1}{m(\xi_0^{n-1}(T)(x,y,s))} &= \log \tfrac{1}{\mu(\gamma_{-m_n(x)(1+o(1))}^{m_n(x)(1+o(1))}(S)(y))} + O(\log n) \\ &= 2m_n(x)h(S)(1 + o(1)) + O(\log n) \\ &\overset{\mathfrak{d}}{=} 2|\mathcal{N}|h(S_1^h)\sqrt{n}(1 + o(1)) \ \ \texttt{as}\ n \to \infty. \quad \square \end{aligned}$$

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